J I a KUMON

Expansion of Polynomial Products

Formula

When m and n are positive integers:

$a^m \times a^n = a^{m+n}$

$(a^m)^n = a^{mn}$

$(ab)^m = a^m b^m$

Note: These formulas are often called the ***properties of powers*** or the ***laws of indices***.

Expand the following expressions.

(1) $(a^3)^4 = a^{3\cdot 4} = \boldsymbol{a^{12}}$

(2) $(-2a^2bc^3)^3 = (-2)^3 a^{2\cdot 3} b^{1\cdot 3} c^{3\cdot 3} = \boldsymbol{-8a^6b^3c^9}$

(3) $3a^2b^3 \times 5a^3b^2 = 3\cdot 5a^{2+3}b^{3+2} = \boldsymbol{15a^5b^5}$

(4) $-4ab\left(-\frac{1}{2}ab^2\right)^2 = -4ab\left[\left(-\frac{1}{2}\right)^2 a^{1\cdot 2}b^{2\cdot 2}\right] = -4ab\left(\frac{1}{4}a^2b^4\right)$

$= -4\cdot\frac{1}{4}a^{1+2}b^{1+4} = \boldsymbol{-a^3b^5}$

J I b

(５)　$-\frac{3}{4}ab\left(-\frac{2}{3}ab^2\right)^2=-\frac{3}{4}ab\left(\frac{4}{9}a^2b^4\right)=\boldsymbol{-\frac{1}{3}a^3b^5}$

(６)　$a^7\div a^5=\frac{a^7}{a^5}=\boldsymbol{a^2}$

(７)　$(a^4)^3\div(a^3)^4=a^{12}\div a^{12}=\frac{a^{12}}{a^{12}}=\boldsymbol{1}$

(８)　$4a^3b\div 3ab^2=\frac{4a^3b}{3ab^2}=\boldsymbol{\frac{4a^2}{3b}}$

(９)　$\frac{2}{3}a^4b^4\times(-3a^2b)\div\frac{1}{4}a^5b^7=\frac{2}{3}a^4b^4\cdot(-3a^2b)\cdot\frac{4}{a^5b^7}=\boldsymbol{-\frac{8a}{b^2}}$

(10)　$$\begin{aligned}\frac{2}{5}x^2y^2\div\frac{4}{15}x^4y^7\times\frac{5}{3}x^2y^3&=\frac{2}{5}x^2y^2\cdot\frac{15}{4x^4y^7}\cdot\frac{5}{3}x^2y^3\\&=\frac{2}{5}\cdot\frac{15}{4}\cdot\frac{5}{3}\cdot\frac{x^{2+2}y^{2+3}}{x^4y^7}\\&=\frac{5x^4y^5}{2x^4y^7}=\boldsymbol{\frac{5}{2y^2}}\end{aligned}$$

J 2a KUMON

Expansion of Polynomial Products

Expand the following expressions.

We can expand brackets as follows:
$a(b+c)=ab+ac$
This is called the ***distributive law***.

(1) $3xy(2yz+4xz)$
$=(3xy)(2yz)+(3xy)(4xz)$
$=\boldsymbol{6xy^2z+12x^2yz}$

(2) $-5yz(-2xz+3xy)$
$=(-5yz)(-2xz)+(-5yz)(3xy)$
$=\boldsymbol{10xyz^2-15xy^2z}$

(3) $4x^2z(-3y^2z+xz^2)$
$=(4x^2z)(-3y^2z)+(4x^2z)(xz^2)$
$=\boldsymbol{-12x^2y^2z^2+4x^3z^3}$

(4) $-2abc(ab+bc+ca)$
$=(-2abc)(ab)+(-2abc)(bc)+(-2abc)(ca)$
$=\boldsymbol{-2a^2b^2c-2ab^2c^2-2a^2bc^2}$

J2b

（5） $9y^2z\left(\frac{2}{3}x^2y-\frac{5}{9}xz^2\right)$

$=(9y^2z)\left(\frac{2}{3}x^2y\right)+(9y^2z)\left(-\frac{5}{9}xz^2\right)$

$=\boldsymbol{6x^2y^3z-5xy^2z^3}$

（6） $8ab^4\left(\frac{a}{2}-\frac{3}{8a^2b^2}\right)$

$=(8ab^4)\left(\frac{a}{2}\right)+(8ab^4)\left(-\frac{3}{8a^2b^2}\right)$

$=\boldsymbol{4a^2b^4-\frac{3b^2}{a}}$

（7） $\frac{1}{2x}(6x^2-8)$

$=\left(\frac{1}{2x}\right)(6x^2)+\left(\frac{1}{2x}\right)(-8)$

$=\boldsymbol{3x-\frac{4}{x}}$

（8） $\frac{1}{3x^2y^2}(9x^3y^3-4x^2y^2)$

$=\left(\frac{1}{3x^2y^2}\right)(9x^3y^3)+\left(\frac{1}{3x^2y^2}\right)(-4x^2y^2)$

$=\boldsymbol{3xy-\frac{4}{3}}$

（9） $(a^3+2b^2x-3cx)\div\frac{2x}{a}$

$=(a^3+2b^2x-3cx)\times\frac{a}{2x}$

$=\left(\frac{a}{2x}\right)(a^3)+\left(\frac{a}{2x}\right)(2b^2x)+\left(\frac{a}{2x}\right)(-3cx)$

$=\boldsymbol{\frac{a^4}{2x}+ab^2-\frac{3ac}{2}}$

J3a KUMON

Expansion of Polynomial Products

Expand the following expressions.

(1) $(a+b)(a-b)$

$=\boldsymbol{a^2-b^2}$

(2) $(-x-2y)(-x+2y)$

$=[(-x)-2y][(-x)+2y]$

$=(-x)^2-(2y)^2$

$=\boldsymbol{x^2-4y^2}$

(3) $(-x-2)^2$

$=(-x)^2-2\cdot(-x)\cdot 2+2^2$

$=\boldsymbol{x^2+4x+4}$

(4) $2(3a^3+4)^2$

$=2(9a^6+24a^3+16)$

$=\boldsymbol{18a^6+48a^3+32}$

(5) $-2(-a+3b)^2$

$=-2(a^2-6ab+9b^2)$

$=\boldsymbol{-2a^2+12ab-18b^2}$

Note: Key formulas from level I :

$(a+b)^2=a^2+2ab+b^2$

$(a-b)^2=a^2-2ab+b^2$

$(a+b)(a-b)=a^2-b^2$

J3b

(6) $\left(x-\frac{1}{2}\right)\left(x+\frac{1}{3}\right)$

$=\boldsymbol{x^2-\frac{1}{6}x-\frac{1}{6}}$

(7) $3(x-2)(x+3)$

$=3(x^2+x-6)$

$=\boldsymbol{3x^2+3x-18}$

(8) $-5(3x+4)(2x-1)$

$=-5(6x^2+5x-4)$

$=\boldsymbol{-30x^2-25x+20}$

(9) $(a+3b)(a+5b)-(a+4b)^2$

$=(a^2+8ab+15b^2)-(a^2+8ab+16b^2)$

$=a^2+8ab+15b^2-a^2-8ab-16b^2$

$=\boldsymbol{-b^2}$

(10) $(3a-7)(2a+1)+(5a-3)(3a+4)$

$=(6a^2-11a-7)+(15a^2+11a-12)$

$=6a^2-11a-7+15a^2+11a-12$

$=\boldsymbol{21a^2-19}$

J 4a

KUMON

Expansion of Polynomial Products

1. Expand the following expressions. Write the intermediate steps.

Ex.

$$(a+b+c)^2$$

Treat $(a+b)$ and c as separate terms.

$$=[(a+b)+c]^2$$

Expand, treating $(a+b)$ as a single unit.

$$=(a+b)^2+2c(a+b)+c^2$$

Expand all $(a+b)$ terms.

$$=a^2+2ab+b^2+2ac+2bc+c^2$$

Rearrange the terms as shown.

$$=a^2+b^2+c^2+2ab+2ac+2bc$$

(1) $(2a-3b+c)^2$

$=[(2a-3b)+c]^2$

$=(2a-3b)^2+2c(2a-3b)+c^2$

$=4a^2-12ab+9b^2+4ac-6bc+c^2$

$=\mathbf{4a^2+9b^2+c^2-12ab+4ac-6bc}$

(2) $(3x-2y-z)^2$

$=[(3x-2y)-z]^2$

$=(3x-2y)^2-2z(3x-2y)+z^2$

$=9x^2-12xy+4y^2-6xz+4yz+z^2$

$=\mathbf{9x^2+4y^2+z^2-12xy-6xz+4yz}$

(3) $(x^2+x-3)^2$

$=[(x^2+x)-3]^2$

$=(x^2+x)^2-6(x^2+x)+9$

$=x^4+2x^3+x^2-6x^2-6x+9$

$=\mathbf{x^4+2x^3-5x^2-6x+9}$

J4b

Formula

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$$

☜ This formula is normally written in this way (cyclical order).

2. Expand using the above formula.

(1) $(2a-3b-c)^2$
$=(2a)^2+(-3b)^2+(-c)^2+2(2a)(-3b)+2(-3b)(-c)+2(-c)(2a)$
$=\boldsymbol{4a^2+9b^2+c^2-12ab+6bc-4ca}$

(2) $(2x-3y+4z)^2=\boldsymbol{4x^2+9y^2+16z^2-12xy-24yz+16zx}$

(3) $(2x+3y+5z)^2=\boldsymbol{4x^2+9y^2+25z^2+12xy+30yz+20zx}$

(4) $(x^2-2x-3)^2=x^4+4x^2+9-4x^3+12x-6x^2$
$=\boldsymbol{x^4-4x^3-2x^2+12x+9}$

(5) $(2x^2-3x+4)^2=4x^4+9x^2+16-12x^3-24x+16x^2$
$=\boldsymbol{4x^4-12x^3+25x^2-24x+16}$

J5a KUMON

Expansion of Polynomial Products

Formula

$(a+b)(a-b)=a^2-b^2$

This formula is called the ***difference of two squares.***

Expand the following expressions.

Ex.

$(a+b+c)(a+b-c)$

$=[(a+b)+c][(a+b)-c]$

Treat $(a+b)$ and c as separate terms.

$=(a+b)^2-c^2$

Expand, treating $(a+b)$ as a single unit.

$=a^2+2ab+b^2-c^2$

(1) $(a+2b+3c)(a+2b-3c)$
$=[(a+2b)+3c][(a+2b)-3c]$
$=(a+2b)^2-(3c)^2$
$=\boldsymbol{a^2+4ab+4b^2-9c^2}$

(2) $(a-3b+5c)(a-3b-5c)$
$=[(a-3b)+5c][(a-3b)-5c]$
$=(a-3b)^2-(5c)^2$
$=\boldsymbol{a^2-6ab+9b^2-25c^2}$

(3) $(a-4b-2c)(a-4b+2c)$
$=[(a-4b)-2c][(a-4b)+2c]$
$=(a-4b)^2-(2c)^2$
$=\boldsymbol{a^2-8ab+16b^2-4c^2}$

J5b

(4) $(a+4b-2c)(a-4b-2c)$

$=[(\ \boldsymbol{a-2c}\ )+4b][(\ \boldsymbol{a-2c}\ )-4b]$

$=(\ \boldsymbol{a-2c}\ )^2-(\ \boldsymbol{4b}\ )^2$

$=\boldsymbol{a^2-4ac+4c^2-16b^2}$

(5) $(a+3b+5c)(a-3b+5c)$

$=[(a+5c)+3b][(a+5c)-3b]$

$=(a+5c)^2-(3b)^2$

$=\boldsymbol{a^2+10ac+25c^2-9b^2}$

(6) $(a+b-c)(a-b+c)$

$=[a+(\ \boldsymbol{b-c}\ )][a-(\ \boldsymbol{b-c}\ )]$

$=a^2-(\ \boldsymbol{b-c}\ )^2$

$=\boldsymbol{a^2-b^2+2bc-c^2}$

(7) $(a+b+c+d)(a+b-c-d)$

$=[(a+b)+(c+d)][(a+b)-(\ \boldsymbol{c+d}\ )]$

$=(a+b)^2-(\ \boldsymbol{c+d}\ )^2$

$=\boldsymbol{a^2+2ab+b^2-c^2-2cd-d^2}$

(8) $(a-b+c+d)(a+b+c-d)$

$=[(a+c)-(b-d)][(a+c)+(b-d)]$

$=(a+c)^2-(b-d)^2$

$=\boldsymbol{a^2+2ac+c^2-b^2+2bd-d^2}$

J 6a KUMON

Expansion of Polynomial Products

Formula

$(x+a)(x+b)=x^2+(a+b)x+ab$

Expand the following expressions.

Ex.

$(x+y+3a)(x+y+5a)$

$=[(x+y)+3a][(x+y)+5a]$ Treat $(x+y)$ as a single unit.

$=(x+y)^2+8a(x+y)+15a^2$

$=x^2+2xy+y^2+8ax+8ay+15a^2$

(1) $(x+2y+3a)(x+2y+4a)$

$=[(x+2y)+3a][(x+2y)+4a]$

$=(x+2y)^2+7a(x+2y)+12a^2$

$=\boldsymbol{x^2+4xy+4y^2+7ax+14ay+12a^2}$

(2) $(2a-4b-c)(2a+2b-c)$

$=[(2a-c)-4b][(2a-c)+2b]$

$=(2a-c)^2-2b(2a-c)-8b^2$

$=\boldsymbol{4a^2-4ac+c^2-4ab+2bc-8b^2}$

J6b

(3) $[5(a+b)-2][3(a+b)+5]$

$=15(a+b)^2+25(a+b)-6(a+b)-10$

$=15(a+b)^2+\boxed{\mathbf{19}}(a+b)-10$

$=\mathbf{15a^2+30ab+15b^2+19a+19b-10}$

(4) $[2(a-b)+3c][3(a-b)-5c]$

$=6(a-b)^2-c(a-b)-15c^2$

$=\mathbf{6a^2-12ab+6b^2-ac+bc-15c^2}$

(5) $(x+2)(x+3)(x+1)(x+4)$

$=(x^2+5x+\boxed{\mathbf{6}})(x^2+5x+\boxed{\mathbf{4}})$

$=[(x^2+5x)+\boxed{\mathbf{6}}][(x^2+5x)+\boxed{\mathbf{4}}]$

$=(x^2+5x)^2+10(x^2+5x)+24$

$=x^4+10x^3+25x^2+10x^2+50x+24$

$=\mathbf{x^4+10x^3+35x^2+50x+24}$

(6) $(x+1)(x+5)(x+2)(x+4)$

$=(x^2+6x+5)(x^2+6x+8)$

$=[(x^2+6x)+5][(x^2+6x)+8]$

$=(x^2+6x)^2+13(x^2+6x)+40$

$=x^4+12x^3+36x^2+13x^2+78x+40$

$=\mathbf{x^4+12x^3+49x^2+78x+40}$

J7a KUMON

Expansion of Polynomial Products

Formula

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

Expand the following expressions.

(1) $(a+1)^3 = a^3+3\cdot a^2\cdot 1+3\cdot a\cdot 1^2+1^3$
$= \boldsymbol{a^3+3a^2+3a+1}$

(2) $(a+2)^3 = \boldsymbol{a^3+6a^2+12a+8}$

(3) $(a+5)^3 = \boldsymbol{a^3+15a^2+75a+125}$

(4) $(2x+1)^3 = (2x)^3+3\cdot(2x)^2\cdot 1+3\cdot 2x\cdot 1^2+1^3$
$= \boldsymbol{8x^3+12x^2+6x+1}$

(5) $(x+2y)^3 = \boldsymbol{x^3+6x^2y+12xy^2+8y^3}$

J7b

Formula

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

(6) $(a-1)^3 = a^3 - 3\cdot a^2\cdot 1 + 3\cdot a\cdot 1^2 - 1^3$
$= \boldsymbol{a^3 - 3a^2 + 3a - 1}$

(7) $(a-3)^3 = \boldsymbol{a^3 - 9a^2 + 27a - 27}$

(8) $(2a-1)^3 = (2a)^3 - 3\cdot(2a)^2\cdot 1 + 3\cdot 2a\cdot 1^2 - 1^3$
$= \boldsymbol{8a^3 - 12a^2 + 6a - 1}$

(9) $(2x-3)^3 = \boldsymbol{8x^3 - 36x^2 + 54x - 27}$

(10) $(x-3y)^3 = \boldsymbol{x^3 - 9x^2y + 27xy^2 - 27y^3}$

J8a KUMON

Expansion of Polynomial Products

Formulas

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a-b)^3 = a^3-3a^2b+3ab^2-b^3$$

Expand the following expressions.

(1) $(a+4)^3 = \boldsymbol{a^3+12a^2+48a+64}$

(2) $(2a-b)^3 = (2a)^3-3\cdot(2a)^2\cdot b+3\cdot 2a\cdot b^2-b^3$
$= \boldsymbol{8a^3-12a^2b+6ab^2-b^3}$

(3) $(3a+2b)^3 = (3a)^3+3\cdot(3a)^2\cdot 2b+3\cdot 3a\cdot(2b)^2+(2b)^3$
$= \boldsymbol{27a^3+54a^2b+36ab^2+8b^3}$

(4) $(x^2+1)^3 = (x^2)^3+3\cdot(x^2)^2\cdot 1+3\cdot x^2\cdot 1^2+1^3$
$= \boldsymbol{x^6+3x^4+3x^2+1}$

(5) $(2x^2-3)^3 = (2x^2)^3-3\cdot(2x^2)^2\cdot 3+3\cdot(2x^2)\cdot 3^2-3^3$
$= \boldsymbol{8x^6-36x^4+54x^2-27}$

J8b

(6) $$(-a-1)^3=[-(a+1)]^3=-(a+1)^3$$
$$=-(a^3+3a^2+3a+1)$$
$$=\boldsymbol{-a^3-3a^2-3a-1}$$

(7) $$(-2a-b)^3=[-(2a+b)]^3=-(2a+b)^3$$
$$=-[(2a)^3+3\cdot(2a)^2\cdot b+3\cdot 2a\cdot b^2+b^3]$$
$$=-(8a^3+12a^2b+6ab^2+b^3)$$
$$=\boldsymbol{-8a^3-12a^2b-6ab^2-b^3}$$

(8) $$(-x^2+3)^3=[-(x^2-3)]^3=-(x^2-3)^3$$
$$=-[(x^2)^3-3\cdot(x^2)^2\cdot 3+3\cdot x^2\cdot 3^2-3^3]$$
$$=-(x^6-9x^4+27x^2-27)$$
$$=\boldsymbol{-x^6+9x^4-27x^2+27}$$

(9) $$\left(a+\frac{1}{3}\right)^3=a^3+3\cdot a^2\cdot\frac{1}{3}+3\cdot a\cdot\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3$$
$$=\boldsymbol{a^3+a^2+\frac{1}{3}a+\frac{1}{27}}$$

(10) $$\left(-2a-\frac{1}{3}\right)^3=\left[-\left(2a+\frac{1}{3}\right)\right]^3=-\left(2a+\frac{1}{3}\right)^3$$
$$=-\left[(2a)^3+3\cdot(2a)^2\cdot\frac{1}{3}+3\cdot 2a\cdot\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3\right]$$
$$=-\left(8a^3+4a^2+\frac{2}{3}a+\frac{1}{27}\right)$$
$$=\boldsymbol{-8a^3-4a^2-\frac{2}{3}a-\frac{1}{27}}$$

J9a KUMON

Expansion of Polynomial Products

Expand as shown in the examples.

Ex.

$$
\begin{aligned}
&(x+1)^3(x-1)^3\\
&=[(x+1)(x-1)]^3\\
&=(x^2-1)^3\\
&=x^6-3x^4+3x^2-1
\end{aligned}
$$

Expand $(x+1)(x-1)$ first.

Note: This is easier than expanding $(x+1)^3$ and $(x-1)^3$.

(1) $(x+2)^3(x-2)^3$

$=[(x+2)(x-2)]^3$

$=(x^2-4)^3$

$=\boldsymbol{x^6-12x^4+48x^2-64}$

Use the formulas on J8a.

(2) $(a+3)^3(a-3)^3$

$=[(a+3)(a-3)]^3$

$=(a^2-9)^3$

$=\boldsymbol{a^6-27a^4+243a^2-729}$

(3) $(x+2y)^3(x-2y)^3$

$=[(x+2y)(x-2y)]^3$

$=(x^2-4y^2)^3$

$=\boldsymbol{x^6-12x^4y^2+48x^2y^4-64y^6}$

J9b

Ex.

$$\begin{aligned}&(x+1)(x+2)(x+3)\\=&(x^2+3x+2)(x+3)\\=&x^3+3x^2+3x^2+9x+2x+6\\=&x^3+6x^2+11x+6\end{aligned}$$

Expand $(x+1)(x+2)$ first.

(4) $(x+1)(x+3)(x+5)$
$=(x^2+4x+3)(x+5)$
$=x^3+5x^2+4x^2+20x+3x+15$
$=\boldsymbol{x^3+9x^2+23x+15}$

(5) $(x-1)(x-2)(x-3)$
$=(x^2-3x+2)(x-3)$
$=x^3-3x^2-3x^2+9x+2x-6$
$=\boldsymbol{x^3-6x^2+11x-6}$

(6) $(x+2)(x-3)(x+4)$
$=(x^2-x-6)(x+4)$
$=x^3+4x^2-x^2-4x-6x-24$
$=\boldsymbol{x^3+3x^2-10x-24}$

(7) $(x+1)(x+2)(x+4)$
$=(x^2+3x+2)(x+4)$
$=x^3+4x^2+3x^2+12x+2x+8$
$=\boldsymbol{x^3+7x^2+14x+8}$

From this, we find

Formula

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$$

J 10a KUMON

Expansion of Polynomial Products

Expand the following expressions.

(1) $\left(3a+\frac{1}{3}\right)^3=\mathbf{27a^3+9a^2+a+\frac{1}{27}}$

Use the formulas on J 8a.

(2) $\left(x-\frac{2}{x}\right)^3=\mathbf{x^3-6x+\frac{12}{x}-\frac{8}{x^3}}$

(3) $\left(x^2+\frac{1}{x}\right)^3=\mathbf{x^6+3x^3+3+\frac{1}{x^3}}$

(4) $$\begin{aligned}(2x+1)^3(2x-1)^3&=[(2x+1)(2x-1)]^3\\&=(4x^2-1)^3\\&=\mathbf{64x^6-48x^4+12x^2-1}\end{aligned}$$

J10b

(5) $(2a-3b+c)^2=\mathbf{4a^2+9b^2+c^2-12ab-6bc+4ca}$

Use the formula on J4b.

(6) $(2x^2-3x-4)^2=4x^4+9x^2+16-12x^3+24x-16x^2$
$=\mathbf{4x^4-12x^3-7x^2+24x+16}$

(7) $(2x^2+3x-4)(2x^2-3x+4)=[2x^2+(3x-4)][2x^2-(3x-4)]$
$=4x^4-(3x-4)^2$
$=\mathbf{4x^4-9x^2+24x-16}$

(8) $(2x^2-x+3)(2x^2+x-3)=[2x^2-(x-3)][2x^2+(x-3)]$
$=4x^4-(x-3)^2$
$=\mathbf{4x^4-x^2+6x-9}$

J11a KUMON

Factorisation I

Formula

$$x^2+(a+b)x+ab=(x+a)(x+b)$$

Factorise the following expressions. (Review of Level I)

(1) $x^2+8x+15=\boldsymbol{(x+5)(x+3)}$

(2) $x^2+10x+21=\boldsymbol{(x+7)(x+3)}$

(3) $a^2-7a+10=\boldsymbol{(a-5)(a-2)}$

(4) $a^2+3ab-10b^2=\boldsymbol{(a+5b)(a-2b)}$

(5) $12x^2+24x-96=12(\ \boldsymbol{x^2+2x-8}\ )$
$=\boldsymbol{12(x+4)(x-2)}$

First take out the common factor 12.

(6) $ax^2-2ax-8a=a(x^2-2x-8)$
$=\boldsymbol{a(x-4)(x+2)}$

J I I b

Ex.

$$(x+y)^2-3(x+y)-10$$
$$=[(x+y)-5][(x+y)+2]$$
$$=(x+y-5)(x+y+2)$$

Treat $(x+y)$ as a single unit.

(7) $(x+y)^2+8(x+y)+15$
$=[(x+y)+5][(x+y)+3]$
$=\boldsymbol{(x+y+5)(x+y+3)}$

(8) $(x+y)^2-(x+y)-12$
$=[(x+y)-4][(x+y)+3]$
$=\boldsymbol{(x+y-4)(x+y+3)}$

(9) $(2x+y)^2+7a(2x+y)+10a^2$
$=[(2x+y)+5a][(2x+y)+2a]$
$=\boldsymbol{(2x+y+5a)(2x+y+2a)}$

(10) $(x-3)^2-(x-3)-42$
$=[(x-3)-7][(x-3)+6]$
$=\boldsymbol{(x-10)(x+3)}$

(11) $(x-5)^2+(x-5)-12$
$=[(x-5)+4][(x-5)-3]$
$=\boldsymbol{(x-1)(x-8)}$

J12a KUMON

Factorisation I

Factorise the following expressions.

(1) $(x+y)^2-5(x+y)+6=[(x+y)-3][(x+y)-2]$
$=\boldsymbol{(x+y-3)(x+y-2)}$

(2) $(x+y)^2-2(x+y)-15=[(x+y)-5][(x+y)+3]$
$=\boldsymbol{(x+y-5)(x+y+3)}$

(3) $(x+y)^2+3(x+y)-18=[(x+y)+6][(x+y)-3]$
$=\boldsymbol{(x+y+6)(x+y-3)}$

(4) $(x+y)^2-7(x+y)+10=[(x+y)-5][(x+y)-2]$
$=\boldsymbol{(x+y-5)(x+y-2)}$

(5) $(x+2)^2-9(x+2)+20=[(x+2)-5][(x+2)-4]$
$=\boldsymbol{(x-3)(x-2)}$

J12b

（6） $(x-y)^2+4z(x-y)+3z^2=[(x-y)+3z][(x-y)+z]$
$=\boldsymbol{(x-y+3z)(x-y+z)}$

（7） $(x+y)^2-2z(x+y)-15z^2=[(x+y)-5z][(x+y)+3z]$
$=\boldsymbol{(x+y-5z)(x+y+3z)}$

（8） $(a+b)^2+6c(a+b)+5c^2=[(a+b)+5c][(a+b)+c]$
$=\boldsymbol{(a+b+5c)(a+b+c)}$

（9） $(a+b)^2-7c(a+b)+10c^2=[(a+b)-5c][(a+b)-2c]$
$=\boldsymbol{(a+b-5c)(a+b-2c)}$

（10） $(a-2b)^2-9c(a-2b)+20c^2=[(a-2b)-5c][(a-2b)-4c]$
$=\boldsymbol{(a-2b-5c)(a-2b-4c)}$

J13a

Factorisation I

Factorise the following expressions.

Ex.

$2x^2+5x-12=(2x-3)(x+4)$

$$\begin{array}{ccc} 2 & \times & -3 \\ 1 & & 4 \end{array}$$

If you can do this mentally, it is not necessary to write this step.

(1) $2x^2+7x+5=\mathbf{(2x+5)(x+1)}$

$$\begin{array}{ccc} 2 & \times & 5 \\ 1 & & 1 \end{array}$$

(2) $6x^2+29x-5=\mathbf{(6x-1)(x+5)}$

$$\begin{array}{ccc} 6 & \times & -1 \\ 1 & & 5 \end{array}$$

(3) $10x^2-7x-12=\mathbf{(5x+4)(2x-3)}$

$$\begin{array}{ccc} 5 & \times & 4 \\ 2 & & -3 \end{array}$$

(4) $6x^2-17x+5=\mathbf{(3x-1)(2x-5)}$

$$\begin{array}{ccc} 3 & \times & -1 \\ 2 & & -5 \end{array}$$

(5) $3x^2-13x+4=\mathbf{(3x-1)(x-4)}$

$$\begin{array}{ccc} 3 & \times & -1 \\ 1 & & -4 \end{array}$$

Further notes on question (1)

— Looking at $2x^2$, write the factors of 2 on the left.

$$\begin{array}{cc} 2 & \times \\ 1 & \end{array}$$

— Looking at $+5$, write the factors of 5 on the right.

$$\begin{array}{ccc} 2 & \times & 5 \\ 1 & & 1 \end{array}$$

— Looking at $+7x$, check these numbers multiply to give $+7$.

$$\begin{array}{ccccc} 2 & \times & 5 & \longrightarrow & 5\times1 \\ 1 & & 1 & \longrightarrow & \underline{2\times1} \\ & & & & 7 \end{array}$$

— It is often necessary to try different combinations of factors and signs.

— Read off the answer directly.

$$\begin{array}{cccl} 2 & \times & 5 & \text{gives } (2x+5) \\ 1 & & 1 & \text{gives } (x+1) \end{array}$$

J13b

(6) $5x^2+16x+3=\boldsymbol{(5x+1)(x+3)}$

$$\begin{array}{ccc} 5 & & 1 \\ & \times & \\ 1 & & 3 \end{array}$$

> You can check in two ways,
>
> (i) multiply out each cross
>
> $$\begin{array}{ccccc} 5 & & 1 & \longrightarrow & 1 \\ & \times & & & \\ 1 & & 3 & \longrightarrow & \underline{15} \\ & & & & 16 \end{array}$$
>
> (ii) expand your final answer
>
> $(5x+1)(x+3)=5x^2+16x+3$

(7) $5x^2-2xy-3y^2=\boldsymbol{(5x+3y)(x-y)}$

$$\begin{array}{ccc} 5 & & 3y \\ & \times & \\ 1 & & -y \end{array}$$

(8) $2x^2+xy-6y^2=\boldsymbol{(2x-3y)(x+2y)}$

$$\begin{array}{ccc} 2 & & -3y \\ & \times & \\ 1 & & 2y \end{array}$$

(9) $4x^2-13x-12=\boldsymbol{(4x+3)(x-4)}$

$$\begin{array}{ccc} 4 & & 3 \\ & \times & \\ 1 & & -4 \end{array}$$

(10) $6x^2-xy-12y^2=\boldsymbol{(3x+4y)(2x-3y)}$

$$\begin{array}{ccc} 3 & & 4y \\ & \times & \\ 2 & & -3y \end{array}$$

J14a KUMON

Factorisation I

Factorise the following expressions.

Ex.

$$2(x+y)^2+7(x+y)+5=[2(x+y)+5][(x+y)+1]$$

Treat $(x+y)$ as a single unit.

$$=(2x+2y+5)(x+y+1)$$

$$\begin{array}{cc} 2 & 5 \\ 1 & 1 \end{array}$$

It is not always necessary to write this step.

(1) $2(x+y)^2+5(x+y)+3=[2(x+y)+3][(x+y)+1]$

$$=\mathbf{(2x+2y+3)(x+y+1)}$$

$$\begin{array}{cc} 2 & 3 \\ 1 & 1 \end{array}$$

(2) $2(x+y)^2+(x+y)-3=[2(x+y)+3][(x+y)-1]$

$$=\mathbf{(2x+2y+3)(x+y-1)}$$

$$\begin{array}{cc} 2 & 3 \\ 1 & -1 \end{array}$$

(3) $2(x+y)^2-9(x+y)-5=[2(x+y)+1][(x+y)-5]$

$$=\mathbf{(2x+2y+1)(x+y-5)}$$

$$\begin{array}{cc} 2 & 1 \\ 1 & -5 \end{array}$$

(4) $3(x+y)^2-13(x+y)+4=[3(x+y)-1][(x+y)-4]$

$$=\mathbf{(3x+3y-1)(x+y-4)}$$

$$\begin{array}{cc} 3 & -1 \\ 1 & -4 \end{array}$$

J 14b

(5) $2(x+y)^2+3(x+y)+1=[2(x+y)+1][(x+y)+1]$
$$=\mathbf{(2x+2y+1)(x+y+1)}$$

$$\begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array}$$

(6) $2(x+y)^2+7a(x+y)+5a^2=[2(x+y)+5a][(x+y)+a]$
$$=\mathbf{(2x+2y+5a)(x+y+a)}$$

$$\begin{array}{cc} 2 & 5a \\ 1 & a \end{array}$$

(7) $3(x-y)^2+2a(x-y)-5a^2=[3(x-y)+5a][(x-y)-a]$
$$=\mathbf{(3x-3y+5a)(x-y-a)}$$

$$\begin{array}{cc} 3 & 5a \\ 1 & -a \end{array}$$

(8) $7(x+y)^2+13a(x+y)-2a^2=[7(x+y)-a][(x+y)+2a]$
$$=\mathbf{(7x+7y-a)(x+y+2a)}$$

$$\begin{array}{cc} 7 & -a \\ 1 & 2a \end{array}$$

J 15a KUMON

Factorisation I

Formulas

$a^2+2ab+b^2=(a+b)^2$

$a^2-2ab+b^2=(a-b)^2$

Factorise the following expressions.

(1) $9x^2-12x+4=\mathbf{(3x-2)^2}$

(2) $x^2+4xy+4y^2=\mathbf{(x+2y)^2}$

(3) $4x^2+12xy+9y^2=\mathbf{(2x+3y)^2}$

(4) $-3ax^2+12axy-12ay^2=-3a(\ \mathbf{x^2-4xy+4y^2}\ )$

$=\mathbf{-3a(x-2y)^2}$

(5) $-x^2+4x-4=-(x^2-4x+4)=\mathbf{-(x-2)^2}$

Take out the common factor -1.

J15b

Ex.

$$(x+y)^2+6(x+y)+9$$
$$=[(x+y)+3]^2$$
$$=(x+y+3)^2$$

Treat $(x+y)$ as a single unit.

(6) $(x+y)^2+4(x+y)+4$
$=[(x+y)+2]^2$
$=\boldsymbol{(x+y+2)^2}$

(7) $(x+y)^2+8(x+y)+16$
$=[(x+y)+4]^2$
$=\boldsymbol{(x+y+4)^2}$

(8) $(x+y)^2-10(x+y)+25$
$=[(x+y)-5]^2$
$=\boldsymbol{(x+y-5)^2}$

(9) $(x-y)^2-6a(x-y)+9a^2$
$=[(x-y)-3a]^2$
$=\boldsymbol{(x-y-3a)^2}$

(10) $(x+2y)^2-2z(x+2y)+z^2$
$=[(x+2y)-z]^2$
$=\boldsymbol{(x+2y-z)^2}$

J16a KUMON

Factorisation I

Factorise the following expressions.

Ex.

$$x^2+2x(a+b)+(a+b)^2$$
$$=[x+(a+b)]^2$$
$$=(x+a+b)^2$$

Treat $(a+b)$ as a single unit.

Use the formulas:
$a^2+2ab+b^2=(a+b)^2$
$a^2-2ab+b^2=(a-b)^2$

(1) $x^2-2x(a-b)+(a-b)^2$
$=[x-(a-b)]^2$
$=\boldsymbol{(x-a+b)^2}$

(2) $x^2+4x(y+z)+4(y+z)^2$
$=[x+2(y+z)]^2$
$=\boldsymbol{(x+2y+2z)^2}$

(3) $a^2+6a(b-c)+9(b-c)^2$
$=[a+3(b-c)]^2$
$=\boldsymbol{(a+3b-3c)^2}$

(4) $a^2-8a(b+c)+16(b+c)^2$
$=[a-4(b+c)]^2$
$=\boldsymbol{(a-4b-4c)^2}$

(5) $a^2-6a(2b-3c)+9(2b-3c)^2$
$=[a-3(2b-3c)]^2$
$=\boldsymbol{(a-6b+9c)^2}$

J16b

(6) $x^2+6x(x+3y)+9(x+3y)^2$

$=[x+3(x+3y)]^2$

$=(x+3x+9y)^2$

$=\boldsymbol{(4x+9y)}^2$

(7) $(x+2y)^2-2y(x+2y)+y^2$

$=[(x+2y)-y]^2$

$=(x+2y-y)^2$

$=\boldsymbol{(x+y)}^2$

(8) $(3x-2y)^2+8y(3x-2y)+16y^2$

$=[(3x-2y)+4y]^2$

$=(3x-2y+4y)^2$

$=\boldsymbol{(3x+2y)}^2$

(9) $(x+a)^2-2(x+a)(y+a)+(y+a)^2$

$=[(x+a)-(y+a)]^2$

$=(x+a-y-a)^2$

$=\boldsymbol{(x-y)}^2$

(10) $(a+b)^2+6(a+b)(2a+b)+9(2a+b)^2$

$=[(a+b)+3(2a+b)]^2$

$=(a+b+6a+3b)^2$

$=\boldsymbol{(7a+4b)}^2$

J 17a

KUMON

Factorisation I

Formula

$$a^2-b^2=(a+b)(a-b)$$

This formula is called the ***difference of two squares.***

Factorise the following expressions.

(1) $9a^2-4b^2=\mathbf{(3a+2b)(3a-2b)}$

(2) $5x^2-20=5(\;\mathbf{x^2-4}\;)$

$=\mathbf{5(x+2)(x-2)}$

The first step is to take out the common factor 5.

(3) $4ax^2-9ay^2=a(4x^2-9y^2)$

$=\mathbf{a(2x+3y)(2x-3y)}$

(4) $(x+y)^2-4=[(x+y)+2][(x+y)-2]$

$=\mathbf{(x+y+2)(x+y-2)}$

(5) $(x+y)^2-z^2=[(x+y)+z][(x+y)-z]$

$=\mathbf{(x+y+z)(x+y-z)}$

J17b

(6) $x^2-(a+b)^2=[x+(a+b)][x-(a+b)]$
$=\boldsymbol{(x+a+b)(x-a-b)}$

(7) $9-(x-2y)^2=[3+(x-2y)][3-(x-2y)]$
$=\boldsymbol{(3+x-2y)(3-x+2y)}$

(8) $x^2-(a-b)^2=[x+(a-b)][x-(a-b)]$
$=\boldsymbol{(x+a-b)(x-a+b)}$

(9) $4a^2-(a-b)^2=[2a+(a-b)][2a-(a-b)]$
$=(2a+a-b)(2a-a+b)$
$=\boldsymbol{(3a-b)(a+b)}$

(10) $(a+b+c)^2-b^2=[(a+b+c)+b][(a+b+c)-b]$
$=(a+b+c+b)(a+b+c-b)$
$=\boldsymbol{(a+2b+c)(a+c)}$

J 18a KUMON

Factorisation I

Factorise the following expressions.

Ex.

$$(a+b)^2-(c+d)^2$$
$$=[(a+b)+(c+d)][(a+b)-(c+d)]$$
$$=(a+b+c+d)(a+b-c-d)$$

Treat $(a+b)$ and $(c+d)$ as single units.

Use the formula for the difference of two squares:
$a^2-b^2=(a+b)(a-b)$

(1) $(a+b)^2-(c-d)^2$
$=[(a+b)+(c-d)][(a+b)-(c-d)]$
$=\boldsymbol{(a+b+c-d)(a+b-c+d)}$

(2) $(a-b)^2-(c-d)^2$
$=[(a-b)+(c-d)][(a-b)-(c-d)]$
$=\boldsymbol{(a-b+c-d)(a-b-c+d)}$

(3) $(a-b)^2-(c+d)^2$
$=[(a-b)+(c+d)][(a-b)-(c+d)]$
$=\boldsymbol{(a-b+c+d)(a-b-c-d)}$

(4) $(x-y)^2-(a-b)^2$
$=[(x-y)+(a-b)][(x-y)-(a-b)]$
$=\boldsymbol{(x-y+a-b)(x-y-a+b)}$

(5) $(3x+2)^2-(x+3)^2$
$=[(3x+2)+(x+3)][(3x+2)-(x+3)]$
$=(3x+2+x+3)(3x+2-x-3)$
$=\boldsymbol{(4x+5)(2x-1)}$

J18b

Ex.

$$(3a+2b)^2-b^2$$
$$=[(3a+2b)+b][(3a+2b)-b]$$
$$=(3a+3b)(3a+b)$$
$$=3(a+b)(3a+b)$$

Factorise completely.

(6) $(3a+2b)^2-(2a+3b)^2$
$=[(3a+2b)+(2a+3b)][(3a+2b)-(2a+3b)]$
$=(3a+2b+2a+3b)(3a+2b-2a-3b)$
$=(5a+5b)(a-b)$
$=\boldsymbol{5(a+b)(a-b)}$

Make sure that your answer is factorised completely.

(7) $(2a-3b)^2-(3a-2b)^2$
$=[(2a-3b)+(3a-2b)][(2a-3b)-(3a-2b)]$
$=(2a-3b+3a-2b)(2a-3b-3a+2b)$
$=(5a-5b)(-a-b)$
$=\boldsymbol{-5(a-b)(a+b)}$

(8) $(x+2y)^2-(2x+y)^2$
$=[(x+2y)+(2x+y)][(x+2y)-(2x+y)]$
$=(x+2y+2x+y)(x+2y-2x-y)$
$=(3x+3y)(-x+y)$
$=\boldsymbol{-3(x+y)(x-y)}$

(9) $(3a-4)^2-(2a-1)^2$
$=[(3a-4)+(2a-1)][(3a-4)-(2a-1)]$
$=(3a-4+2a-1)(3a-4-2a+1)$
$=(5a-5)(a-3)$
$=\boldsymbol{5(a-1)(a-3)}$

(10) $(2x-5)^2-9$
$=[(2x-5)+3][(2x-5)-3]$
$=(2x-5+3)(2x-5-3)$
$=(2x-2)(2x-8)$
$=\boldsymbol{4(x-1)(x-4)}$

J 19a KUMON

Factorisation I

Factorise the following expressions.

Ex.

$$x^4 - y^4$$
$$= (x^2)^2 - (y^2)^2$$
Treat x^2 and y^2 as single units.
$$= (x^2 + y^2)(x^2 - y^2)$$
Factorise completely.
$$= (x^2 + y^2)(x + y)(x - y)$$

(1) $x^4 - 16$
$$= (x^2)^2 - 4^2$$
$$= (x^2 + 4)(x^2 - 4)$$
$$= \mathbf{(x^2 + 4)(x + 2)(x - 2)}$$

(2) $16x^4 - y^4$
$$= (4x^2)^2 - (y^2)^2$$
$$= (4x^2 + y^2)(4x^2 - y^2)$$
$$= \mathbf{(4x^2 + y^2)(2x + y)(2x - y)}$$

(3) $x^8 - y^8$
$$= (x^4)^2 - (y^4)^2$$
$$= (x^4 + y^4)(x^4 - y^4)$$
$$= (x^4 + y^4)[(x^2)^2 - (y^2)^2]$$
$$= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2)$$
$$= \mathbf{(x^4 + y^4)(x^2 + y^2)(x + y)(x - y)}$$

(4) $x^8 - 1$
$$= (x^4)^2 - 1$$
$$= (x^4 + 1)(x^4 - 1)$$
$$= (x^4 + 1)[(x^2)^2 - 1]$$
$$= (x^4 + 1)(x^2 + 1)(x^2 - 1)$$
$$= \mathbf{(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)}$$

J19b

(5) x^4-5x^2-36

$=(\ \boldsymbol{x^2-9}\ )(x^2+4)$

$=\boldsymbol{(x+3)(x-3)(x^2+4)}$

(6) x^4-8x^2-9

$=(x^2-9)(x^2+1)$

$=\boldsymbol{(x+3)(x-3)(x^2+1)}$

(7) x^4-6x^2+8

$=(x^2-4)(x^2-2)$

$=\boldsymbol{(x+2)(x-2)(x^2-2)}$

It is possible to write this as $(x+2)(x-2)(x+\sqrt{2})(x-\sqrt{2})$. However, this is not normally done unless solving an equation.

(8) $2x^4+x^2-3$

$=(2x^2+3)(x^2-1)$

$=\boldsymbol{(2x^2+3)(x+1)(x-1)}$

(9) ax^4-3ax^2-4a

$=a(x^4-3x^2-4)$

$=a(x^2-4)(x^2+1)$

$=\boldsymbol{a(x+2)(x-2)(x^2+1)}$

J 20a

KUMON

Factorisation I

Factorise the following expressions.

Ex.

$$
\begin{aligned}
&x^4-2x^2+1\\
&=(x^2-1)^2\\
&=[(x+1)(x-1)]^2\\
&=(x+1)^2(x-1)^2
\end{aligned}
$$

Factorise inside the brackets.

(1) x^4-8x^2+16

$$
\begin{aligned}
&=(x^2-4)^2\\
&=[(x+2)(x-2)]^2\\
&=\boldsymbol{(x+2)^2(x-2)^2}
\end{aligned}
$$

(2) x^4-18x^2+81

$$
\begin{aligned}
&=(x^2-9)^2\\
&=[(x+3)(x-3)]^2\\
&=\boldsymbol{(x+3)^2(x-3)^2}
\end{aligned}
$$

(3) $x^4-2x^2y^2+y^4$

$$
\begin{aligned}
&=(x^2-y^2)^2\\
&=[(x+y)(x-y)]^2\\
&=\boldsymbol{(x+y)^2(x-y)^2}
\end{aligned}
$$

(4) x^5-8x^3+16x

$$
\begin{aligned}
&=x(x^4-8x^2+16)\\
&=x(x^2-4)^2\\
&=x[(x+2)(x-2)]^2\\
&=\boldsymbol{x(x+2)^2(x-2)^2}
\end{aligned}
$$

J 20b

(5) $x^4 - a^4$

$= (x^2)^2 - (a^2)^2$

$= (x^2 + a^2)(x^2 - a^2)$

$= \boldsymbol{(x^2 + a^2)(x + a)(x - a)}$

(6) $x^7 - 16x^3$

$= x^3(x^4 - 16)$

$= x^3[(x^2)^2 - 4^2]$

$= x^3(x^2 + 4)(x^2 - 4)$

$= \boldsymbol{x^3(x^2 + 4)(x + 2)(x - 2)}$

(7) $x^4 - 8x^2y^2 + 16y^4$

$= (x^2 - 4y^2)^2$

$= [(x + 2y)(x - 2y)]^2$

$= \boldsymbol{(x + 2y)^2(x - 2y)^2}$

Note Summary

- For 1. $(x+y)^2 - 3(x+y) - 10 =$
  and 2. $(a+b)^2 - (c+d)^2 =$
  treat $(x+y)$, $(a+b)$ and $(c+d)$ as single units.
- To factorise
  $(x+y)^2 - 3(x+y) - 10$,
  the following methods can be used:

  Method (i) Let $x + y = A$,
  and substitute to get
  $A^2 - 3A - 10$

  Method (ii) Underline or circle the $(x+y)$ terms
  $(\underline{x+y})^2 - 3(\underline{x+y}) - 10$
  $(x+y)^2 - 3(x+y) - 10$
  and treat these as single units.

J21a KUMON

Factorisation II

Factorise the following expressions.

Ex.

$x(a-2b)+y(a-2b)=(a-2b)(x+y)$

$2a(x+y)+4(x+y)=2(x+y)(a+2)$ Take out the greatest common factor, $2(x+y)$.

(1) $x(2a-b)-y(2a-b)=\mathbf{(2a-b)(x-y)}$

(2) $3x(a+1)+6(a+1)=\mathbf{3(a+1)(x+2)}$

The greatest common factor is $3(a+1)$.

(3) $4a(x+3)+6(x+3)=\mathbf{2(x+3)(2a+3)}$

(4) $3a(x-3)+3(x-3)=\mathbf{3(x-3)(a+1)}$

(5) $3x(a-2)-6(a-2)=\mathbf{3(a-2)(x-2)}$

J21b

Ex.

$$a(x-3)+b(3-x)$$
$$=a(x-3)-b(x-3)$$
$$=(x-3)(a-b)$$

Write $(3-x)$ as $-(x-3)$, then factorise.

(6) $2(x-2)+a(2-x)$

$=2(x-2)-a(x-2)$

$=\boldsymbol{(x-2)(2-a)}$

$[=\boldsymbol{-(x-2)(a-2)}]$

(7) $x(a-b)-2(b-a)$

$=x(a-b)+2(a-b)$

$=\boldsymbol{(a-b)(x+2)}$

(8) $5x(2a-b)+2y(b-2a)$

$=5x(2a-b)-2y(2a-b)$

$=\boldsymbol{(2a-b)(5x-2y)}$

(9) $3ab(2x-y)-5a^2(y-2x)$

$=3ab(2x-y)+5a^2(2x-y)$

$=\boldsymbol{a(2x-y)(3b+5a)}$

$[=\boldsymbol{a(2x-y)(5a+3b)}]$

(10) $2b(x-2y)-4a(2y-x)$

$=2b(x-2y)+4a(x-2y)$

$=\boldsymbol{2(x-2y)(b+2a)}$

$[=\boldsymbol{2(x-2y)(2a+b)}]$

J 22a

KUMON

Factorisation II

Factorise the following expressions.

Ex.

$$x^2(a-2)+y^2(2-a)$$
$$=x^2(a-2)-y^2(a-2)$$
$$=(a-2)(x^2-y^2)$$
$$=(a-2)(x+y)(x-y)$$

$(2-a)=-(a-2)$

Factorise (x^2-y^2).

(1) $9x^2(a-b)+4(b-a)$
$=9x^2(a-b)-4(a-b)$
$=(a-b)(9x^2-4)$
$=\boldsymbol{(a-b)(3x+2)(3x-2)}$

(2) $x^2(a-b)+16(b-a)$
$=x^2(a-b)-16(a-b)$
$=(a-b)(x^2-16)$
$=\boldsymbol{(a-b)(x+4)(x-4)}$

(3) $4x^2(x-2y)+9y^2(2y-x)$
$=4x^2(x-2y)-9y^2(x-2y)$
$=(x-2y)(4x^2-9y^2)$
$=\boldsymbol{(x-2y)(2x+3y)(2x-3y)}$

(4) $a^2(x-2y)+4b^2(2y-x)$
$=a^2(x-2y)-4b^2(x-2y)$
$=(x-2y)(a^2-4b^2)$
$=\boldsymbol{(x-2y)(a+2b)(a-2b)}$

(5) $-a^2(3y-x)-9(x-3y)$
$=a^2(x-3y)-9(x-3y)$
$=(x-3y)(a^2-9)$
$=\boldsymbol{(x-3y)(a+3)(a-3)}$

J22b

Ex.

$$
\begin{aligned}
&x^2(a-b)+x(a-b)+2(b-a) \\
&=x^2(a-b)+x(a-b)-2(a-b) \\
&=(a-b)(x^2+x-2) \\
&=(a-b)(x+2)(x-1)
\end{aligned}
$$

$(b-a)=-(a-b)$

(6) $x^2(a-b)+x(b-a)+2(b-a)$
$=x^2(a-b)-x(a-b)-2(a-b)$
$=(a-b)(x^2-x-2)$
$=\boldsymbol{(a-b)(x-2)(x+1)}$

(7) $x^2(a-b)-6x(a-b)-9(b-a)$
$=x^2(a-b)-6x(a-b)+9(a-b)$
$=(a-b)(x^2-6x+9)$
$=\boldsymbol{(a-b)(x-3)^2}$

Use the formula:
$a^2-2ab+b^2=(a-b)^2$

(8) $2x^2(a-b)+x(b-a)+6(b-a)$
$=2x^2(a-b)-x(a-b)-6(a-b)$
$=(a-b)(2x^2-x-6)$
$=\boldsymbol{(a-b)(2x+3)(x-2)}$

(9) $(a-b)(x^2-5)+(b-a)(3x+5)$
$=(a-b)(x^2-5)-(a-b)(3x+5)$
$=(a-b)(x^2-5-3x-5)$
$=(a-b)(x^2-3x-10)$
$=\boldsymbol{(a-b)(x-5)(x+2)}$

(10) $(a-b)(2x^2+9)-(b-a)(7x-6)$
$=(a-b)(2x^2+9)+(a-b)(7x-6)$
$=(a-b)(2x^2+9+7x-6)$
$=(a-b)(2x^2+7x+3)$
$=\boldsymbol{(a-b)(2x+1)(x+3)}$

J 23a KUMON

Factorisation II

Factorise the following expressions.

Ex.

$$2(a-b)+(b-a)^2$$
$$=2(a-b)+(a-b)^2$$
$$=(a-b)[2+(a-b)]$$
$$=(a-b)(2+a-b)$$

$(b-a)^2=(a-b)^2$
The sign of the term does not change.

Note: $(b-a)^2=[-(a-b)]^2$
$=(-1)^2(a-b)^2$
$=(a-b)^2$

The answer can also be written $(a-b)(a-b+2)$.

(1) $3(a-b)+(b-a)^2$
$=3(a-b)+(a-b)^2$
$=(a-b)[3+(a-b)]$
$=\boldsymbol{(a-b)(3+a-b)}$

(2) $3(a-b)-(b-a)^2$
$=3(a-b)-(a-b)^2$
$=(a-b)[3-(a-b)]$
$=\boldsymbol{(a-b)(3-a+b)}$

(3) $(y-x)^2+3(x-y)$
$=(x-y)^2+3(x-y)$
$=(x-y)[(x-y)+3]$
$=\boldsymbol{(x-y)(x-y+3)}$

[Why the sign does not change.]
If we compare
$$(b-a)^2=b^2-2ba+a^2$$
$$(a-b)^2=a^2-2ab+b^2$$
both give the same result.

J23b

(4) $2(2y-x)^2+4a(x-2y)$

$=2(x-2y)^2+4a(x-2y)$

$=2(x-2y)[(x-2y)+2a]$

$=\boldsymbol{2(x-2y)(x-2y+2a)}$

(5) $a(2y-x)^2+2a^2(x-2y)$

$=a(x-2y)^2+2a^2(x-2y)$

$=a(x-2y)[(x-2y)+2a]$

$=\boldsymbol{a(x-2y)(x-2y+2a)}$

(6) $x(3b-2a)^2-2x^2(2a-3b)$

$=x(2a-3b)^2-2x^2(2a-3b)$

$=x(2a-3b)[(2a-3b)-2x]$

$=\boldsymbol{x(2a-3b)(2a-3b-2x)}$

(7) $2(x-y)-(y-x)^2$

$=2(x-y)-(x-y)^2$

$=(x-y)[2-(x-y)]$

$=\boldsymbol{(x-y)(2-x+y)}$

(8) $2a^2(x-3y)+a(3y-x)^2$

$=2a^2(x-3y)+a(x-3y)^2$

$=a(x-3y)[2a+(x-3y)]$

$=\boldsymbol{a(x-3y)(2a+x-3y)}$

J 24 a

KUMON

Factorisation II

Factorise the following expressions.

Ex.

$$
\begin{aligned}
& xy(x-y)+3y(y-x)^2 \\
&= xy(x-y)+3y(x-y)^2 \\
&= y(x-y)[x+3(x-y)] \\
&= y(x-y)(4x-3y)
\end{aligned}
$$

$(y-x)^2=(x-y)^2$

(1) $2x(x-y)-3(y-x)^2$

$$
\begin{aligned}
&= 2x(x-y)-3(x-y)^2 \\
&= (x-y)[2x-3(x-y)] \\
&= (x-y)(2x-3x+3y) \\
&= \boldsymbol{(x-y)(-x+3y)}
\end{aligned}
$$

$[= \boldsymbol{-(x-y)(x-3y)}]$

(2) $x(2y-x)^2+2x^2(x-2y)$

$$
\begin{aligned}
&= x(x-2y)^2+2x^2(x-2y) \\
&= x(x-2y)[(x-2y)+2x] \\
&= x(x-2y)(x-2y+2x) \\
&= \boldsymbol{x(x-2y)(3x-2y)}
\end{aligned}
$$

(3) $4x^2(x-3y)-2x(3y-x)^2$

$$
\begin{aligned}
&= 4x^2(x-3y)-2x(x-3y)^2 \\
&= 2x(x-3y)[2x-(x-3y)] \\
&= 2x(x-3y)(2x-x+3y) \\
&= \boldsymbol{2x(x-3y)(x+3y)}
\end{aligned}
$$

(4) $xy(x-2y)+3y(2y-x)^2$

$$
\begin{aligned}
&= xy(x-2y)+3y(x-2y)^2 \\
&= y(x-2y)[x+3(x-2y)] \\
&= y(x-2y)(x+3x-6y) \\
&= y(x-2y)(4x-6y) \\
&= \boldsymbol{2y(x-2y)(2x-3y)}
\end{aligned}
$$

J24b

(5) $2a^2(a-3b)+a(3b-a)^2$

$=2a^2(a-3b)+a(a-3b)^2$

$=a(a-3b)[2a+(a-3b)]$

$=a(a-3b)(2a+a-3b)$

$=a(a-3b)(3a-3b)$

$=\boldsymbol{3a(a-3b)(a-b)}$

(6) $(a-b)^2(3x-5y)-(b-a)^2(x-y)$

$=(a-b)^2(3x-5y)-(a-b)^2(x-y)$

$=(a-b)^2[(3x-5y)-(x-y)]$

$=(a-b)^2(3x-5y-x+y)$

$=(a-b)^2(2x-4y)$

$=\boldsymbol{2(a-b)^2(x-2y)}$

(7) $3x^2y(x-y)-6xy^2(y-x)^2$

$=3x^2y(x-y)-6xy^2(x-y)^2$

$=3xy(x-y)[x-2y(x-y)]$

$=\boldsymbol{3xy(x-y)(x-2xy+2y^2)}$

(8) $5xy^2(2y-3x)-15x^2(3x-2y)^2$

$=5xy^2(2y-3x)-15x^2(2y-3x)^2$

$=5x(2y-3x)[y^2-3x(2y-3x)]$

$=5x(2y-3x)(y^2-6xy+9x^2)$

$=\boldsymbol{5x(2y-3x)(y-3x)^2}$

$[=\boldsymbol{-5x(3x-2y)(3x-y)^2}]$

J 25a

KUMON

Factorisation II

Factorise the following expressions.

Ex.

$$2(x-y)^2 + (y-x)^3$$

$(y-x)^3 = -(x-y)^3$
The sign of the term changes.

$$= 2(x-y)^2 - (x-y)^3$$

$$= (x-y)^2[2-(x-y)]$$

$$= (x-y)^2(2-x+y)$$

Note: $(y-x)^3 = [-(x-y)]^3 = (-1)^3(x-y)^3 = -(x-y)^3$

(1) $(x-y)^2 + 3(y-x)^3$
$= (x-y)^2 - 3(x-y)^3$
$= (x-y)^2[1-3(x-y)]$
$= \boldsymbol{(x-y)^2(1-3x+3y)}$

(2) $x(a-b)^2 + 2y(b-a)^3$
$= x(a-b)^2 - 2y(a-b)^3$
$= (a-b)^2[x-2y(a-b)]$
$= \boldsymbol{(a-b)^2(x-2ay+2by)}$

(3) $4a^2(x-3)^2 - 2(3-x)^3$
$= 4a^2(x-3)^2 + 2(x-3)^3$
$= 2(x-3)^2[2a^2+(x-3)]$
$= \boldsymbol{2(x-3)^2(2a^2+x-3)}$

(4) $x^2y(x-3y)^2 + xy^2(3y-x)^3$
$= x^2y(x-3y)^2 - xy^2(x-3y)^3$
$= xy(x-3y)^2[x-y(x-3y)]$
$= \boldsymbol{xy(x-3y)^2(x-xy+3y^2)}$

J25b

(5) $6xy^2(3y-x)-3x^2(x-3y)^2$
$=6xy^2(3y-x)-3x^2(3y-x)^2$
$=3x(3y-x)[2y^2-x(3y-x)]$
$=3x(3y-x)(2y^2-3xy+x^2)$
$=\mathbf{3x(3y-x)(2y-x)(y-x)}$
$[=\mathbf{-3x(x-3y)(x-2y)(x-y)}]$

(6) $5x^2y(x-y)^3-10xy^3(y-x)^2$
$=5x^2y(x-y)^3-10xy^3(x-y)^2$
$=5xy(x-y)^2[x(x-y)-2y^2]$
$=5xy(x-y)^2(x^2-xy-2y^2)$
$=\mathbf{5xy(x-y)^2(x-2y)(x+y)}$

(7) $2x^2(x-2y)^2+6xy^2(2y-x)$
$=2x^2(2y-x)^2+6xy^2(2y-x)$
$=2x(2y-x)[x(2y-x)+3y^2]$
$=2x(2y-x)(3y^2+2xy-x^2)$
$=\mathbf{2x(2y-x)(3y-x)(y+x)}$
$[=\mathbf{2x(x-2y)(x-3y)(x+y)}]$

Note Summary

A. 1. When the power is an even number, the sign of the term remains the same:

$(b-a)^2=(a-b)^2, \quad (b-a)^4=(a-b)^4$

2. When the power is an odd number, the sign of the term changes:

$(b-a)=-(a-b), \quad (b-a)^3=-(a-b)^3$

B. There is often a choice in how to rewrite:

1. $a(x-3y)^2-b(3y-x)^3=a(x-3y)^2+b\mathbf{(x-3y)}^3$

2. $a(x-3y)^2-b(3y-x)^3=a\mathbf{(3y-x)}^2-b(3y-x)^3$

Either way is correct.

J 26a KUMON

Factorisation II

Factorise the following expressions.

Ex.

$$ax+ay+bx+by$$
$$=a(x+y)+b(x+y)$$
$$=(x+y)(a+b)$$

ax and ay have a in common.
bx and by have b in common.

(1) $ax-ay+bx-by$
$=a(x-y)+b(x-y)$
$=\boldsymbol{(x-y)(a+b)}$

(2) $ax+ay-bx-by$
$=a(x+y)-b(x+y)$
$=\boldsymbol{(x+y)(a-b)}$

(3) $ax-ay-bx+by$
$=a(x-y)-b(x-y)$
$=\boldsymbol{(x-y)(a-b)}$

(4) $ab+ac+bd+cd$
$=a(b+c)+d(b+c)$
$=\boldsymbol{(b+c)(a+d)}$

(5) $ab-ac+bd-cd$
$=a(b-c)+d(b-c)$
$=\boldsymbol{(b-c)(a+d)}$

Note: In the example, we could get the same result by grouping the expression into x and y terms.

$$ax+ay+bx+by=x(a+b)+y(a+b)=(a+b)(x+y)$$

J 26b

(6) $ab+cd-ac-bd$

$=ab-ac-bd+cd$

$=a(b-c)-d(b-c)$

$=\boldsymbol{(b-c)(a-d)}$

(7) $ab+cd+bc+ad$

$=ab+ad+bc+cd$

$=a(b+d)+c(b+d)$

$=\boldsymbol{(b+d)(a+c)}$

(8) $ab+cd-bd-ac$

$=ab-bd-ac+cd$

$=b(a-d)-c(a-d)$

$=\boldsymbol{(a-d)(b-c)}$

(9) $ab-cd+bd-ac$

$=ab+bd-ac-cd$

$=b(a+d)-c(a+d)$

$=\boldsymbol{(a+d)(b-c)}$

(10) $ab-cd-bd+ac$

$=ab-bd+ac-cd$

$=b(a-d)+c(a-d)$

$=\boldsymbol{(a-d)(b+c)}$

J 27a

KUMON

Factorisation II

Factorise the following expressions.

(1) $a^2+xy+ax+ay$
$=a^2+ax+ay+xy$
$=a(a+x)+y(a+x)$
$=\boldsymbol{(a+x)(a+y)}$

(2) $a^2-xy+ax-ay$
$=a^2+ax-ay-xy$
$=a(a+x)-y(a+x)$
$=\boldsymbol{(a+x)(a-y)}$

(3) $a^2-xy-ax+ay$
$=a^2-ax+ay-xy$
$=a(a-x)+y(a-x)$
$=\boldsymbol{(a-x)(a+y)}$

(4) $2ax+3by+3bx+2ay$
$=2ax+2ay+3bx+3by$
$=2a(x+y)+3b(x+y)$
$=\boldsymbol{(x+y)(2a+3b)}$

(5) $2ax-3by+3bx-2ay$
$=2ax-2ay+3bx-3by$
$=2a(x-y)+3b(x-y)$
$=\boldsymbol{(x-y)(2a+3b)}$

J27b

(6) $a^2+2ax+ab+2bx$
$=a(a+2x)+b(a+2x)$
$=\boldsymbol{(a+2x)(a+b)}$

(7) $a^2+2ax-ab-2bx$
$=a(a+2x)-b(a+2x)$
$=\boldsymbol{(a+2x)(a-b)}$

(8) $2ax-ab-a^2+2bx$
$=2ax+2bx-a^2-ab$
$=2x(a+b)-a(a+b)$
$=\boldsymbol{(a+b)(2x-a)}$

(9) $3ax+2bx+3ay+2by$
$=3ax+3ay+2bx+2by$
$=3a(x+y)+2b(x+y)$
$=\boldsymbol{(x+y)(3a+2b)}$

(10) $2ax-3by-6bx+ay$
$=2ax+ay-6bx-3by$
$=a(2x+y)-3b(2x+y)$
$=\boldsymbol{(2x+y)(a-3b)}$

J 28a KUMON

Factorisation II

Factorise the following expressions.

Ex.

$$x^3+4x^2+2x+8$$
$$=x^2(x+4)+2(x+4)$$
$$=(x+4)(x^2+2)$$

x^3 and $4x^2$ have x^2 in common.
$2x$ and 8 have 2 in common.

(1) x^3-4x^2+2x-8
$=x^2(x-4)+2(x-4)$
$=\boldsymbol{(x-4)(x^2+2)}$

(2) x^3+4x^2-2x-8
$=x^2(x+4)-2(x+4)$
$=\boldsymbol{(x+4)(x^2-2)}$

(3) $x^3-2x-4x^2+8$
$=x(x^2-2)-4(x^2-2)$
$=\boldsymbol{(x^2-2)(x-4)}$

(4) x^3+x^2+x+1
$=x^2(x+1)+(x+1)$
$=\boldsymbol{(x+1)(x^2+1)}$

(5) x^3-x^2+x-1
$=x^2(x-1)+(x-1)$
$=\boldsymbol{(x-1)(x^2+1)}$

J28b

(6) x^3-2x^2+x-2

$= x^2(x-2)+(x-2)$

$= \boldsymbol{(x-2)(x^2+1)}$

(7) x^3+x^2-4x-4

$= x^2(x+1)-4(x+1)$

$= (x+1)(x^2-4)$

$= \boldsymbol{(x+1)(x+2)(x-2)}$

Remember to factorise completely.

(8) x^3-x-x^2+1

$= x(x^2-1)-(x^2-1)$

$= (x^2-1)(x-1)$

$= \boldsymbol{(x+1)(x-1)^2}$

(9) $x^2y^2-x^2+y^2-1$

$= x^2(y^2-1)+(y^2-1)$

$= (y^2-1)(x^2+1)$

$= \boldsymbol{(y+1)(y-1)(x^2+1)}$

(10) $x^2y^2-x^2-y^2+1$

$= x^2(y^2-1)-(y^2-1)$

$= (y^2-1)(x^2-1)$

$= \boldsymbol{(y+1)(y-1)(x+1)(x-1)}$

J 29a

KUMON

Factorisation II

Factorise the following expressions.

(1) $x^2 - xy - x + y$

$= x(x-y) - (x-y)$

$= \mathbf{(x-y)(x-1)}$

(2) $x^3 + x^2y - xy^2 - y^3$

$= x^2(x+y) - y^2(x+y)$

$= (x+y)(x^2 - y^2)$

$= (x+y)(x+y)(x-y)$

$= \mathbf{(x+y)^2(x-y)}$

(3) $xy + 1 + x + y$

$= xy + x + y + 1$

$= x(y+1) + (y+1)$

$= \mathbf{(y+1)(x+1)}$

(4) $x^2y + y^2z + x^2z + y^3$

$= x^2y + x^2z + y^3 + y^2z$

$= x^2(y+z) + y^2(y+z)$

$= \mathbf{(y+z)(x^2+y^2)}$

(5) $1 - x - x^2 + x^3$

$= x^3 - x^2 - x + 1$

$= x^2(x-1) - (x-1)$

$= (x-1)(x^2-1)$

$= (x-1)(x+1)(x-1)$

$= \mathbf{(x-1)^2(x+1)}$

J29b

Ex.

$$a^2+2ab+b^2+ax+bx$$
$$=(a+b)^2+x(a+b)$$
$$=(a+b)[(a+b)+x]$$
$$=(a+b)(a+b+x)$$

$a^2+2ab+b^2=(a+b)^2$

(6) $a^2-2ab+b^2-ax+bx$
$=(a-b)^2-x(a-b)$
$=(a-b)[(a-b)-x]$
$=\boldsymbol{(a-b)(a-b-x)}$

(7) $ab-ac-b^2+2bc-c^2$
$=a(b-c)-(b^2-2bc+c^2)$
$=a(b-c)-(b-c)^2$
$=(b-c)[a-(b-c)]$
$=\boldsymbol{(b-c)(a-b+c)}$

(8) $a^2-b^2-ac+bc$
$=(a+b)(a-b)-c(a-b)$
$=(a-b)[(a+b)-c]$
$=\boldsymbol{(a-b)(a+b-c)}$

(9) $ab+ac-b^2-2bc-c^2$
$=a(b+c)-(b^2+2bc+c^2)$
$=a(b+c)-(b+c)^2$
$=(b+c)[a-(b+c)]$
$=\boldsymbol{(b+c)(a-b-c)}$

J 30 a

KUMON

Factorisation II

Factorise the following expressions.

(1) $x(2y-x)^2+2x^2(x-2y)$
$=x(x-2y)^2+2x^2(x-2y)$
$=x(x-2y)[(x-2y)+2x]$
$=x(x-2y)(x-2y+2x)$
$=\boldsymbol{x(x-2y)(3x-2y)}$

(2) $xy(x-2y)+3y(2y-x)^2$
$=xy(x-2y)+3y(x-2y)^2$
$=y(x-2y)[x+3(x-2y)]$
$=y(x-2y)(x+3x-6y)$
$=y(x-2y)(4x-6y)$
$=\boldsymbol{2y(x-2y)(2x-3y)}$

(3) $(a-2b)(3x-5y)+(2b-a)(x-y)$
$=(a-2b)(3x-5y)-(a-2b)(x-y)$
$=(a-2b)[(3x-5y)-(x-y)]$
$=(a-2b)(3x-5y-x+y)$
$=(a-2b)(2x-4y)$
$=\boldsymbol{2(a-2b)(x-2y)}$

(4) $4xy^2(3y-x)-2x^2(x-3y)^2$
$=4xy^2(3y-x)-2x^2(3y-x)^2$
$=2x(3y-x)[2y^2-x(3y-x)]$
$=2x(3y-x)(2y^2-3xy+x^2)$
$=\boldsymbol{2x(3y-x)(2y-x)(y-x)}$
$[=\boldsymbol{-2x(x-3y)(x-2y)(x-y)}]$

(5) $5xy^2(2y-3x)-15x^2(3x-2y)^2$
$=5xy^2(2y-3x)-15x^2(2y-3x)^2$
$=5x(2y-3x)[y^2-3x(2y-3x)]$
$=5x(2y-3x)(y^2-6xy+9x^2)$
$=\boldsymbol{5x(2y-3x)(y-3x)^2}$
$[=\boldsymbol{-5x(3x-2y)(3x-y)^2}]$

(6) $2xy-4z^2-2xz+4yz$

$=2(xy-xz+2yz-2z^2)$

$=2[x(y-z)+2z(y-z)]$

$=\mathbf{2(y-z)(x+2z)}$

(7) $6-9x^2+12y-18x^2y$

$=3(2-3x^2+4y-6x^2y)$

$=3[(2-3x^2)+2y(2-3x^2)]$

$=\mathbf{3(2-3x^2)(1+2y)}$

(8) $ax^2y^2-ax^2-ay^2+a$

$=a(x^2y^2-x^2-y^2+1)$

$=a[x^2(y^2-1)-(y^2-1)]$

$=a(y^2-1)(x^2-1)$

$=\mathbf{a(y+1)(y-1)(x+1)(x-1)}$

Note Summary

A. Group like terms to get common factors.

For example, for $ax+ay+bx+by$,

a can be taken from $ax+ay$ to give $a(x+y)$

and b can be taken from $bx+by$ to give $b(x+y)$.

Then we can see that $(x+y)$ is the common factor.

B. We can group using a and b:

$$ax+ay+bx+by=a(x+y)+b(x+y)=(x+y)(a+b)$$

or using x and y:

$$ax+ay+bx+by=x(a+b)+y(a+b)=(a+b)(x+y)$$

Either way is correct.

J 2 1 a KUMON

Factorisation II

Factorise the following expressions.

Ex.

$x(a-2b)+y(a-2b)=(a-2b)(x+y)$

$2a(x+y)+4(x+y)=2(x+y)(a+2)$ ☜ Take out the greatest common factor, $2(x+y)$.

(1) $x(2a-b)-y(2a-b)=\boldsymbol{(2a-b)(x-y)}$

(2) $3x(a+1)+6(a+1)=\boldsymbol{3(a+1)(x+2)}$

The greatest common factor is $3(a+1)$.

(3) $4a(x+3)+6(x+3)=\boldsymbol{2(x+3)(2a+3)}$

(4) $3a(x-3)+3(x-3)=\boldsymbol{3(x-3)(a+1)}$

(5) $3x(a-2)-6(a-2)=\boldsymbol{3(a-2)(x-2)}$

J21b

Ex.

$$a(x-3)+b(\underset{\sim\sim\sim}{3-x})$$
$$=a(x-3)-b(\underset{\sim\sim\sim}{x-3})$$
$$=(x-3)(a-b)$$

Write $(3-x)$ as $-(x-3)$, then factorise.

(6) $2(x-2)+a(2-x)$
$=2(x-2)-a(x-2)$
$=\boldsymbol{(x-2)(2-a)}$
$[=\boldsymbol{-(x-2)(a-2)}]$

(7) $x(a-b)-2(b-a)$
$=x(a-b)+2(a-b)$
$=\boldsymbol{(a-b)(x+2)}$

(8) $5x(2a-b)+2y(b-2a)$
$=5x(2a-b)-2y(2a-b)$
$=\boldsymbol{(2a-b)(5x-2y)}$

(9) $3ab(2x-y)-5a^2(y-2x)$
$=3ab(2x-y)+5a^2(2x-y)$
$=\boldsymbol{a(2x-y)(3b+5a)}$
$[=\boldsymbol{a(2x-y)(5a+3b)}]$

(10) $2b(x-2y)-4a(2y-x)$
$=2b(x-2y)+4a(x-2y)$
$=\boldsymbol{2(x-2y)(b+2a)}$
$[=\boldsymbol{2(x-2y)(2a+b)}]$

J 22a

KUMON

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned} & x^2(a-2)+y^2(2-a) \\ =\ & x^2(a-2)-y^2(a-2) \\ =\ & (a-2)(x^2-y^2) \\ =\ & (a-2)(x+y)(x-y) \end{aligned}$$

$(2-a)=-(a-2)$

Factorise (x^2-y^2).

(1) $9x^2(a-b)+4(b-a)$
$=9x^2(a-b)-4(a-b)$
$=(a-b)(9x^2-4)$
$=\boldsymbol{(a-b)(3x+2)(3x-2)}$

(2) $x^2(a-b)+16(b-a)$
$=x^2(a-b)-16(a-b)$
$=(a-b)(x^2-16)$
$=\boldsymbol{(a-b)(x+4)(x-4)}$

(3) $4x^2(x-2y)+9y^2(2y-x)$
$=4x^2(x-2y)-9y^2(x-2y)$
$=(x-2y)(4x^2-9y^2)$
$=\boldsymbol{(x-2y)(2x+3y)(2x-3y)}$

(4) $a^2(x-2y)+4b^2(2y-x)$
$=a^2(x-2y)-4b^2(x-2y)$
$=(x-2y)(a^2-4b^2)$
$=\boldsymbol{(x-2y)(a+2b)(a-2b)}$

(5) $-a^2(3y-x)-9(x-3y)$
$=a^2(x-3y)-9(x-3y)$
$=(x-3y)(a^2-9)$
$=\boldsymbol{(x-3y)(a+3)(a-3)}$

J22b

Ex.

$$\begin{aligned}&x^2(a-b)+x(a-b)+2(b-a)\\=&x^2(a-b)+x(a-b)-2(a-b)\\=&(a-b)(x^2+x-2)\\=&(a-b)(x+2)(x-1)\end{aligned}$$

$(b-a)=-(a-b)$

(6) $x^2(a-b)+x(b-a)+2(b-a)$
$=x^2(a-b)-x(a-b)-2(a-b)$
$=(a-b)(x^2-x-2)$
$=\boldsymbol{(a-b)(x-2)(x+1)}$

(7) $x^2(a-b)-6x(a-b)-9(b-a)$
$=x^2(a-b)-6x(a-b)+9(a-b)$
$=(a-b)(x^2-6x+9)$
$=\boldsymbol{(a-b)(x-3)^2}$

Use the formula:
$a^2-2ab+b^2=(a-b)^2$

(8) $2x^2(a-b)+x(b-a)+6(b-a)$
$=2x^2(a-b)-x(a-b)-6(a-b)$
$=(a-b)(2x^2-x-6)$
$=\boldsymbol{(a-b)(2x+3)(x-2)}$

(9) $(a-b)(x^2-5)+(b-a)(3x+5)$
$=(a-b)(x^2-5)-(a-b)(3x+5)$
$=(a-b)(x^2-5-3x-5)$
$=(a-b)(x^2-3x-10)$
$=\boldsymbol{(a-b)(x-5)(x+2)}$

(10) $(a-b)(2x^2+9)-(b-a)(7x-6)$
$=(a-b)(2x^2+9)+(a-b)(7x-6)$
$=(a-b)(2x^2+9+7x-6)$
$=(a-b)(2x^2+7x+3)$
$=\boldsymbol{(a-b)(2x+1)(x+3)}$

J 23a KUMON

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned} & 2(a-b)\underset{\sim}{+}(b-a)^2 \\ =\ & 2(a-b)\underset{\sim}{+}(a-b)^2 \\ =\ & (a-b)[2+(a-b)] \\ =\ & (a-b)(2+a-b) \end{aligned}$$

$(b-a)^2=(a-b)^2$
The sign of the term does not change.

Note: $(b-a)^2=[-(a-b)]^2$
$=(-1)^2(a-b)^2$
$=(a-b)^2$

The answer can also be written $(a-b)(a-b+2)$.

(1) $3(a-b)+(b-a)^2$
$=3(a-b)+(a-b)^2$
$=(a-b)[3+(a-b)]$
$=\boldsymbol{(a-b)(3+a-b)}$

(2) $3(a-b)-(b-a)^2$
$=3(a-b)-(a-b)^2$
$=(a-b)[3-(a-b)]$
$=\boldsymbol{(a-b)(3-a+b)}$

(3) $(y-x)^2+3(x-y)$
$=(x-y)^2+3(x-y)$
$=(x-y)[(x-y)+3]$
$=\boldsymbol{(x-y)(x-y+3)}$

[Why the sign does not change.]
If we compare

$(b-a)^2=b^2-2ba+a^2$
$(a-b)^2=a^2-2ab+b^2$

both give the same result.

J23b

（4） $2(2y-x)^2+4a(x-2y)$
$=2(x-2y)^2+4a(x-2y)$
$=2(x-2y)[(x-2y)+2a]$
$=\mathbf{2(x-2y)(x-2y+2a)}$

（5） $a(2y-x)^2+2a^2(x-2y)$
$=a(x-2y)^2+2a^2(x-2y)$
$=a(x-2y)[(x-2y)+2a]$
$=\mathbf{a(x-2y)(x-2y+2a)}$

（6） $x(3b-2a)^2-2x^2(2a-3b)$
$=x(2a-3b)^2-2x^2(2a-3b)$
$=x(2a-3b)[(2a-3b)-2x]$
$=\mathbf{x(2a-3b)(2a-3b-2x)}$

（7） $2(x-y)-(y-x)^2$
$=2(x-y)-(x-y)^2$
$=(x-y)[2-(x-y)]$
$=\mathbf{(x-y)(2-x+y)}$

（8） $2a^2(x-3y)+a(3y-x)^2$
$=2a^2(x-3y)+a(x-3y)^2$
$=a(x-3y)[2a+(x-3y)]$
$=\mathbf{a(x-3y)(2a+x-3y)}$

J 24 a

KUMON

Factorisation II

Factorise the following expressions.

Ex.

$$xy(x-y)+3y(y-x)^2$$
$$=xy(x-y)+3y(x-y)^2 \quad (y-x)^2=(x-y)^2$$
$$=y(x-y)[x+3(x-y)]$$
$$=y(x-y)(4x-3y)$$

(1) $2x(x-y)-3(y-x)^2$
$=2x(x-y)-3(x-y)^2$
$=(x-y)[2x-3(x-y)]$
$=(x-y)(2x-3x+3y)$
$=\boldsymbol{(x-y)(-x+3y)}$
$[=\boldsymbol{-(x-y)(x-3y)}]$

(2) $x(2y-x)^2+2x^2(x-2y)$
$=x(x-2y)^2+2x^2(x-2y)$
$=x(x-2y)[(x-2y)+2x]$
$=x(x-2y)(x-2y+2x)$
$=\boldsymbol{x(x-2y)(3x-2y)}$

(3) $4x^2(x-3y)-2x(3y-x)^2$
$=4x^2(x-3y)-2x(x-3y)^2$
$=2x(x-3y)[2x-(x-3y)]$
$=2x(x-3y)(2x-x+3y)$
$=\boldsymbol{2x(x-3y)(x+3y)}$

(4) $xy(x-2y)+3y(2y-x)^2$
$=xy(x-2y)+3y(x-2y)^2$
$=y(x-2y)[x+3(x-2y)]$
$=y(x-2y)(x+3x-6y)$
$=y(x-2y)(4x-6y)$
$=\boldsymbol{2y(x-2y)(2x-3y)}$

J24b

(5) $2a^2(a-3b)+a(3b-a)^2$
$=2a^2(a-3b)+a(a-3b)^2$
$=a(a-3b)[2a+(a-3b)]$
$=a(a-3b)(2a+a-3b)$
$=a(a-3b)(3a-3b)$
$=\boldsymbol{3a(a-3b)(a-b)}$

(6) $(a-b)^2(3x-5y)-(b-a)^2(x-y)$
$=(a-b)^2(3x-5y)-(a-b)^2(x-y)$
$=(a-b)^2[(3x-5y)-(x-y)]$
$=(a-b)^2(3x-5y-x+y)$
$=(a-b)^2(2x-4y)$
$=\boldsymbol{2(a-b)^2(x-2y)}$

(7) $3x^2y(x-y)-6xy^2(y-x)^2$
$=3x^2y(x-y)-6xy^2(x-y)^2$
$=3xy(x-y)[x-2y(x-y)]$
$=\boldsymbol{3xy(x-y)(x-2xy+2y^2)}$

(8) $5xy^2(2y-3x)-15x^2(3x-2y)^2$
$=5xy^2(2y-3x)-15x^2(2y-3x)^2$
$=5x(2y-3x)[y^2-3x(2y-3x)]$
$=5x(2y-3x)(y^2-6xy+9x^2)$
$=\boldsymbol{5x(2y-3x)(y-3x)^2}$
$[=\boldsymbol{-5x(3x-2y)(3x-y)^2}]$

J 25a KUMON

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned} & 2(x-y)^2 \underset{\sim}{+} (y-x)^3 \\ = & 2(x-y)^2 \underset{\sim}{-} (x-y)^3 \\ = & (x-y)^2[2-(x-y)] \\ = & (x-y)^2(2-x+y) \end{aligned}$$

$(y-x)^3=-(x-y)^3$
The sign of the term changes.

Note: $(y-x)^3=[-(x-y)]^3=(-1)^3(x-y)^3=-(x-y)^3$

(1) $(x-y)^2+3(y-x)^3$
$=(x-y)^2-3(x-y)^3$
$=(x-y)^2[1-3(x-y)]$
$=\boldsymbol{(x-y)^2(1-3x+3y)}$

(2) $x(a-b)^2+2y(b-a)^3$
$=x(a-b)^2-2y(a-b)^3$
$=(a-b)^2[x-2y(a-b)]$
$=\boldsymbol{(a-b)^2(x-2ay+2by)}$

(3) $4a^2(x-3)^2-2(3-x)^3$
$=4a^2(x-3)^2+2(x-3)^3$
$=2(x-3)^2[2a^2+(x-3)]$
$=\boldsymbol{2(x-3)^2(2a^2+x-3)}$

(4) $x^2y(x-3y)^2+xy^2(3y-x)^3$
$=x^2y(x-3y)^2-xy^2(x-3y)^3$
$=xy(x-3y)^2[x-y(x-3y)]$
$=\boldsymbol{xy(x-3y)^2(x-xy+3y^2)}$

(5) $6xy^2(3y-x)-3x^2(x-3y)^2$
$=6xy^2(3y-x)-3x^2(3y-x)^2$
$=3x(3y-x)[2y^2-x(3y-x)]$
$=3x(3y-x)(2y^2-3xy+x^2)$
$=\boldsymbol{3x(3y-x)(2y-x)(y-x)}$
$[=\boldsymbol{-3x(x-3y)(x-2y)(x-y)}]$

(6) $5x^2y(x-y)^3-10xy^3(y-x)^2$
$=5x^2y(x-y)^3-10xy^3(x-y)^2$
$=5xy(x-y)^2[x(x-y)-2y^2]$
$=5xy(x-y)^2(x^2-xy-2y^2)$
$=\boldsymbol{5xy(x-y)^2(x-2y)(x+y)}$

(7) $2x^2(x-2y)^2+6xy^2(2y-x)$
$=2x^2(2y-x)^2+6xy^2(2y-x)$
$=2x(2y-x)[x(2y-x)+3y^2]$
$=2x(2y-x)(3y^2+2xy-x^2)$
$=\boldsymbol{2x(2y-x)(3y-x)(y+x)}$
$[=\boldsymbol{2x(x-2y)(x-3y)(x+y)}]$

Note Summary

A. 1. When the power is an even number, the sign of the term remains the same:

$(b-a)^2=(a-b)^2, \quad (b-a)^4=(a-b)^4$

2. When the power is an odd number, the sign of the term changes:

$(b-a)=-(a-b), \quad (b-a)^3=-(a-b)^3$

B. There is often a choice in how to rewrite:

1. $a(x-3y)^2-b(3y-x)^3=a(x-3y)^2+b\boldsymbol{(x-3y)}^3$

2. $a(x-3y)^2-b(3y-x)^3=a\boldsymbol{(3y-x)}^2-b(3y-x)^3$

Either way is correct.

J 26a KUMON

Factorisation II

Factorise the following expressions.

Ex.

$$ax+ay+bx+by$$
$$=a(x+y)+b(x+y)$$
$$=(x+y)(a+b)$$

ax and ay have a in common.
bx and by have b in common.

(1) $ax-ay+bx-by$
$=a(x-y)+b(x-y)$
$=\boldsymbol{(x-y)(a+b)}$

(2) $ax+ay-bx-by$
$=a(x+y)-b(x+y)$
$=\boldsymbol{(x+y)(a-b)}$

(3) $ax-ay-bx+by$
$=a(x-y)-b(x-y)$
$=\boldsymbol{(x-y)(a-b)}$

(4) $ab+ac+bd+cd$
$=a(b+c)+d(b+c)$
$=\boldsymbol{(b+c)(a+d)}$

(5) $ab-ac+bd-cd$
$=a(b-c)+d(b-c)$
$=\boldsymbol{(b-c)(a+d)}$

Note: In the example, we could get the same result by grouping the expression into x and y terms.

$$ax+ay+bx+by=x(a+b)+y(a+b)=(a+b)(x+y)$$

J26b

（6） $ab+cd-ac-bd$

$=ab-ac-bd+cd$

$=a(b-c)-d(b-c)$

$=\boldsymbol{(b-c)(a-d)}$

（7） $ab+cd+bc+ad$

$=ab+ad+bc+cd$

$=a(b+d)+c(b+d)$

$=\boldsymbol{(b+d)(a+c)}$

（8） $ab+cd-bd-ac$

$=ab-bd-ac+cd$

$=b(a-d)-c(a-d)$

$=\boldsymbol{(a-d)(b-c)}$

（9） $ab-cd+bd-ac$

$=ab+bd-ac-cd$

$=b(a+d)-c(a+d)$

$=\boldsymbol{(a+d)(b-c)}$

(10) $ab-cd-bd+ac$

$=ab-bd+ac-cd$

$=b(a-d)+c(a-d)$

$=\boldsymbol{(a-d)(b+c)}$

J 27a

KUMON

Factorisation II

Factorise the following expressions.

(1) $a^2+xy+ax+ay$
$=a^2+ax+ay+xy$
$=a(a+x)+y(a+x)$
$=\boldsymbol{(a+x)(a+y)}$

(2) $a^2-xy+ax-ay$
$=a^2+ax-ay-xy$
$=a(a+x)-y(a+x)$
$=\boldsymbol{(a+x)(a-y)}$

(3) $a^2-xy-ax+ay$
$=a^2-ax+ay-xy$
$=a(a-x)+y(a-x)$
$=\boldsymbol{(a-x)(a+y)}$

(4) $2ax+3by+3bx+2ay$
$=2ax+2ay+3bx+3by$
$=2a(x+y)+3b(x+y)$
$=\boldsymbol{(x+y)(2a+3b)}$

(5) $2ax-3by+3bx-2ay$
$=2ax-2ay+3bx-3by$
$=2a(x-y)+3b(x-y)$
$=\boldsymbol{(x-y)(2a+3b)}$

J27b

(6) $a^2+2ax+ab+2bx$

$=a(a+2x)+b(a+2x)$

$=\boldsymbol{(a+2x)(a+b)}$

(7) $a^2+2ax-ab-2bx$

$=a(a+2x)-b(a+2x)$

$=\boldsymbol{(a+2x)(a-b)}$

(8) $2ax-ab-a^2+2bx$

$=2ax+2bx-a^2-ab$

$=2x(a+b)-a(a+b)$

$=\boldsymbol{(a+b)(2x-a)}$

(9) $3ax+2bx+3ay+2by$

$=3ax+3ay+2bx+2by$

$=3a(x+y)+2b(x+y)$

$=\boldsymbol{(x+y)(3a+2b)}$

(10) $2ax-3by-6bx+ay$

$=2ax+ay-6bx-3by$

$=a(2x+y)-3b(2x+y)$

$=\boldsymbol{(2x+y)(a-3b)}$

J 28a KUMON

Factorisation II

Factorise the following expressions.

Ex.

$$x^3+4x^2+2x+8$$
$$=x^2(x+4)+2(x+4)$$
$$=(x+4)(x^2+2)$$

x^3 and $4x^2$ have x^2 in common.
$2x$ and 8 have 2 in common.

(1) x^3-4x^2+2x-8
$=x^2(x-4)+2(x-4)$
$=\mathbf{(x-4)(x^2+2)}$

(2) x^3+4x^2-2x-8
$=x^2(x+4)-2(x+4)$
$=\mathbf{(x+4)(x^2-2)}$

(3) $x^3-2x-4x^2+8$
$=x(x^2-2)-4(x^2-2)$
$=\mathbf{(x^2-2)(x-4)}$

(4) x^3+x^2+x+1
$=x^2(x+1)+(x+1)$
$=\mathbf{(x+1)(x^2+1)}$

(5) x^3-x^2+x-1
$=x^2(x-1)+(x-1)$
$=\mathbf{(x-1)(x^2+1)}$

J28b

(6) x^3-2x^2+x-2

$=x^2(x-2)+(x-2)$

$=\boldsymbol{(x-2)(x^2+1)}$

(7) x^3+x^2-4x-4

$=x^2(x+1)-4(x+1)$

$=(x+1)(x^2-4)$

Remember to factorise completely.

$=\boldsymbol{(x+1)(x+2)(x-2)}$

(8) x^3-x-x^2+1

$=x(x^2-1)-(x^2-1)$

$=(x^2-1)(x-1)$

$=\boldsymbol{(x+1)(x-1)^2}$

(9) $x^2y^2-x^2+y^2-1$

$=x^2(y^2-1)+(y^2-1)$

$=(y^2-1)(x^2+1)$

$=\boldsymbol{(y+1)(y-1)(x^2+1)}$

(10) $x^2y^2-x^2-y^2+1$

$=x^2(y^2-1)-(y^2-1)$

$=(y^2-1)(x^2-1)$

$=\boldsymbol{(y+1)(y-1)(x+1)(x-1)}$

J 29a KUMON

Factorisation II

Factorise the following expressions.

(1) $x^2-xy-x+y$
$=x(x-y)-(x-y)$
$=\boldsymbol{(x-y)(x-1)}$

(2) $x^3+x^2y-xy^2-y^3$
$=x^2(x+y)-y^2(x+y)$
$=(x+y)(x^2-y^2)$
$=(x+y)(x+y)(x-y)$
$=\boldsymbol{(x+y)^2(x-y)}$

(3) $xy+1+x+y$
$=xy+x+y+1$
$=x(y+1)+(y+1)$
$=\boldsymbol{(y+1)(x+1)}$

(4) $x^2y+y^2z+x^2z+y^3$
$=x^2y+x^2z+y^3+y^2z$
$=x^2(y+z)+y^2(y+z)$
$=\boldsymbol{(y+z)(x^2+y^2)}$

(5) $1-x-x^2+x^3$
$=x^3-x^2-x+1$
$=x^2(x-1)-(x-1)$
$=(x-1)(x^2-1)$
$=(x-1)(x+1)(x-1)$
$=\boldsymbol{(x-1)^2(x+1)}$

J29b

Ex.

$$
\begin{aligned}
& a^2+2ab+b^2+ax+bx \\
=& (a+b)^2+x(a+b) \\
=& (a+b)[(a+b)+x] \\
=& (a+b)(a+b+x)
\end{aligned}
$$

$a^2+2ab+b^2=(a+b)^2$

(6) $a^2-2ab+b^2-ax+bx$
$=(a-b)^2-x(a-b)$
$=(a-b)[(a-b)-x]$
$=\boldsymbol{(a-b)(a-b-x)}$

(7) $ab-ac-b^2+2bc-c^2$
$=a(b-c)-(b^2-2bc+c^2)$
$=a(b-c)-(b-c)^2$
$=(b-c)[a-(b-c)]$
$=\boldsymbol{(b-c)(a-b+c)}$

(8) $a^2-b^2-ac+bc$
$=(a+b)(a-b)-c(a-b)$
$=(a-b)[(a+b)-c]$
$=\boldsymbol{(a-b)(a+b-c)}$

(9) $ab+ac-b^2-2bc-c^2$
$=a(b+c)-(b^2+2bc+c^2)$
$=a(b+c)-(b+c)^2$
$=(b+c)[a-(b+c)]$
$=\boldsymbol{(b+c)(a-b-c)}$

J 30a KUMON

Factorisation II

Factorise the following expressions.

(1) $x(2y-x)^2+2x^2(x-2y)$
$=x(x-2y)^2+2x^2(x-2y)$
$=x(x-2y)[(x-2y)+2x]$
$=x(x-2y)(x-2y+2x)$
$=\boldsymbol{x(x-2y)(3x-2y)}$

(2) $xy(x-2y)+3y(2y-x)^2$
$=xy(x-2y)+3y(x-2y)^2$
$=y(x-2y)[x+3(x-2y)]$
$=y(x-2y)(x+3x-6y)$
$=y(x-2y)(4x-6y)$
$=\boldsymbol{2y(x-2y)(2x-3y)}$

(3) $(a-2b)(3x-5y)+(2b-a)(x-y)$
$=(a-2b)(3x-5y)-(a-2b)(x-y)$
$=(a-2b)[(3x-5y)-(x-y)]$
$=(a-2b)(3x-5y-x+y)$
$=(a-2b)(2x-4y)$
$=\boldsymbol{2(a-2b)(x-2y)}$

(4) $4xy^2(3y-x)-2x^2(x-3y)^2$
$=4xy^2(3y-x)-2x^2(3y-x)^2$
$=2x(3y-x)[2y^2-x(3y-x)]$
$=2x(3y-x)(2y^2-3xy+x^2)$
$=\boldsymbol{2x(3y-x)(2y-x)(y-x)}$
$[=\boldsymbol{-2x(x-3y)(x-2y)(x-y)}]$

(5) $5xy^2(2y-3x)-15x^2(3x-2y)^2$
$=5xy^2(2y-3x)-15x^2(2y-3x)^2$
$=5x(2y-3x)[y^2-3x(2y-3x)]$
$=5x(2y-3x)(y^2-6xy+9x^2)$
$=\boldsymbol{5x(2y-3x)(y-3x)^2}$
$[=\boldsymbol{-5x(3x-2y)(3x-y)^2}]$

J30b

(6) $2xy-4z^2-2xz+4yz$
$=2(xy-xz+2yz-2z^2)$
$=2[x(y-z)+2z(y-z)]$
$=\mathbf{2(y-z)(x+2z)}$

(7) $6-9x^2+12y-18x^2y$
$=3(2-3x^2+4y-6x^2y)$
$=3[(2-3x^2)+2y(2-3x^2)]$
$=\mathbf{3(2-3x^2)(1+2y)}$

(8) $ax^2y^2-ax^2-ay^2+a$
$=a(x^2y^2-x^2-y^2+1)$
$=a[x^2(y^2-1)-(y^2-1)]$
$=a(y^2-1)(x^2-1)$
$=\mathbf{a(y+1)(y-1)(x+1)(x-1)}$

Note Summary

A. Group like terms to get common factors.

For example, for $ax+ay+bx+by$,

a can be taken from $ax+ay$ to give $a(x+y)$

and b can be taken from $bx+by$ to give $b(x+y)$.

Then we can see that $(x+y)$ is the common factor.

B. We can group using a and b:

$$ax+ay+bx+by=a(x+y)+b(x+y)=(x+y)(a+b)$$

or using x and y:

$$ax+ay+bx+by=x(a+b)+y(a+b)=(a+b)(x+y)$$

Either way is correct.

J31a KUMON

Factorisation III

Factorise the following expressions.

Ex.

$$x^2+(2a+b)x+2ab$$
$$=(x+2a)(x+b)$$

Look for two terms that multiply to $2ab$ and add to $(2a+b)$.
The terms are $2a$ and b.

(1) $x^2+(a+3b)x+3ab$
$=\boldsymbol{(x+a)(x+3b)}$

(2) $x^2-(2a+b)x+2ab$
$=\boldsymbol{(x-2a)(x-b)}$

(3) $x^2+(2a-b)x-2ab$
$=\boldsymbol{(x+2a)(x-b)}$

(4) $x^2-(2a-b)x-2ab$
$=\boldsymbol{(x-2a)(x+b)}$

(5) $x^2-(a-3b)x-3ab$
$=\boldsymbol{(x-a)(x+3b)}$

J31b

Ex.

$$x^2+2x-a(a+2)$$
$$=(x-a)[x+(a+2)]$$
$$=(x-a)(x+a+2)$$

Consider which factor, a or $(a+2)$, will take the negative sign, $[-]$.

$$\begin{bmatrix} a-(a+2)=-2 \\ -a+(a+2)=2 \end{bmatrix}$$

(6) $x^2+3x-a(a+3)$
$=(x-a)[x+(a+3)]$
$=\boldsymbol{(x-a)(x+a+3)}$

(7) $x^2+bx-a(a+b)$
$=(x-a)[x+(a+b)]$
$=\boldsymbol{(x-a)(x+a+b)}$

(8) $x^2-bx-a(a+b)$
$=(x+a)[x-(a+b)]$
$=\boldsymbol{(x+a)(x-a-b)}$

(9) $x^2+2x-a(a-2)$
$=(x+a)[x-(a-2)]$
$=\boldsymbol{(x+a)(x-a+2)}$

J 32a

KUMON

Factorisation III

Factorise the following expressions.

Ex.

$$x^2+(2y+5)x+(y+6)(y-1)$$
$$=[x+(y+6)][x+(y-1)]$$
$$=(x+y+6)(x+y-1)$$

When multiplied: $(y+6)(y-1)$
When added: $(2y+5)$

(1) $x^2+(3y+4)x+(2y+3)(y+1)$
$=[x+(2y+3)][x+(y+1)]$
$=\boldsymbol{(x+2y+3)(x+y+1)}$

(2) $x^2+(3y+5)x+(2y+3)(y+2)$
$=[x+(2y+3)][x+(y+2)]$
$=\boldsymbol{(x+2y+3)(x+y+2)}$

(3) $x^2-(2y+5)x+(y+6)(y-1)$
$=[x-(y+6)][x-(y-1)]$
$=\boldsymbol{(x-y-6)(x-y+1)}$

(4) $x^2-(3y+4)x+(2y+3)(y+1)$
$=[x-(2y+3)][x-(y+1)]$
$=\boldsymbol{(x-2y-3)(x-y-1)}$

J 32 b

Ex.

$$x^2+(y+4)x-(2y+1)(3y+5)$$
$$=[x-(2y+1)][x+(3y+5)]$$
$$=(x-2y-1)(x+3y+5)$$

Consider which factor, $(2y+1)$ or $(3y+5)$, will take the negative sign, $[-]$.

$$\begin{bmatrix}(2y+1)-(3y+5)=-y-4\\ -(2y+1)+(3y+5)=y+4\end{bmatrix}$$

(5) $x^2+(y-7)x-(y+6)(2y-1)$
$=[x-(y+6)][x+(2y-1)]$
$=\boldsymbol{(x-y-6)(x+2y-1)}$

(6) $x^2-(y+2)x-(2y+3)(y+1)$
$=[x-(2y+3)][x+(y+1)]$
$=\boldsymbol{(x-2y-3)(x+y+1)}$

(7) $x^2+(y+1)x-(2y+3)(y+2)$
$=[x+(2y+3)][x-(y+2)]$
$=\boldsymbol{(x+2y+3)(x-y-2)}$

(8) $x^2-(y-4)x-(2y-1)(3y-5)$
$=[x+(2y-1)][x-(3y-5)]$
$=\boldsymbol{(x+2y-1)(x-3y+5)}$

J 33a KUMON

Factorisation III

Factorise the following expressions.

Ex.

$$x^2+(3y+4)x+(2y^2+5y+3)$$
$$=x^2+(3y+4)x+(2y+3)(y+1)$$
$$=[x+(2y+3)][x+(y+1)]$$
$$=(x+2y+3)(x+y+1)$$

Factorise $(2y^2+5y+3)$ first.

(1) $x^2+(3y+5)x+(2y^2+7y+6)$
$=x^2+(3y+5)x+(2y+3)(y+2)$
$=[x+(2y+3)][x+(y+2)]$
$=\boldsymbol{(x+2y+3)(x+y+2)}$

Factorise the "non-x" terms first.

(2) $x^2-(3y+4)x+(2y^2+3y-5)$
$=x^2-(3y+4)x+(2y+5)(y-1)$
$=[x-(2y+5)][x-(y-1)]$
$=\boldsymbol{(x-2y-5)(x-y+1)}$

(3) $x^2-(5y-6)x+(6y^2-13y+5)$
$=x^2-(5y-6)x+(3y-5)(2y-1)$
$=[x-(3y-5)][x-(2y-1)]$
$=\boldsymbol{(x-3y+5)(x-2y+1)}$

J33b

(4) $x^2+yx-(6y^2-5y+1)$

$=x^2+yx-(3y-1)(2y-1)$

$=[x+(3y-1)][x-(2y-1)]$

$=\boldsymbol{(x+3y-1)(x-2y+1)}$

(5) $x^2-(y+4)x-(2y^2+y-3)$

$=x^2-(y+4)x-(2y+3)(y-1)$

$=[x-(2y+3)][x+(y-1)]$

$=\boldsymbol{(x-2y-3)(x+y-1)}$

(6) $x^2-(y-1)x-(2y^2+11y+12)$

$=x^2-(y-1)x-(2y+3)(y+4)$

$=[x-(2y+3)][x+(y+4)]$

$=\boldsymbol{(x-2y-3)(x+y+4)}$

(7) $x^2-(2y+1)x-(3y^2-11y+6)$

$=x^2-(2y+1)x-(3y-2)(y-3)$

$=[x-(3y-2)][x+(y-3)]$

$=\boldsymbol{(x-3y+2)(x+y-3)}$

(8) $x^2-yx-(6y^2-5y+1)$

$=x^2-yx-(3y-1)(2y-1)$

$=[x-(3y-1)][x+(2y-1)]$

$=\boldsymbol{(x-3y+1)(x+2y-1)}$

J 34 a KUMON

Factorisation III

Factorise the following expressions.

Ex.

$$\begin{aligned}
&x^2+5xy+6x+6y^2+13y+5\\
&=x^2+(5y+6)x+(6y^2+13y+5)\\
&=x^2+(5y+6)x+(3y+5)(2y+1)\\
&=[x+(3y+5)][x+(2y+1)]\\
&=(x+3y+5)(x+2y+1)
\end{aligned}$$

Arrange in descending powers of x.

Factorise the "non-x" terms first.

(1) $x^2+3xy+2y^2+5x+7y+6$

$=x^2+(3y+5)x+(2y^2+7y+6)$

$=x^2+(3y+5)x+(2y+3)(y+2)$

$=[x+(2y+3)][x+(y+2)]$

$=\boldsymbol{(x+2y+3)(x+y+2)}$

(2) $x^2-3xy+2y^2-5x+7y+6$

$=x^2-(3y+5)x+(2y^2+7y+6)$

$=x^2-(3y+5)x+(2y+3)(y+2)$

$=[x-(2y+3)][x-(y+2)]$

$=\boldsymbol{(x-2y-3)(x-y-2)}$

(3) $x^2+3xy+2y^2-5x-7y+6$

$=x^2+(3y-5)x+(2y^2-7y+6)$

$=x^2+(3y-5)x+(2y-3)(y-2)$

$=[x+(2y-3)][x+(y-2)]$

$=\boldsymbol{(x+2y-3)(x+y-2)}$

J34b

(4) $x^2-5xy+6y^2+6x-13y+5$
$=x^2-(5y-6)x+(6y^2-13y+5)$
$=x^2-(5y-6)x+(3y-5)(2y-1)$
$=[x-(3y-5)][x-(2y-1)]$
$=\boldsymbol{(x-3y+5)(x-2y+1)}$

(5) $x^2+2y^2+3+3xy+4x+5y$
$=x^2+(3y+4)x+(2y^2+5y+3)$
$=x^2+(3y+4)x+(2y+3)(y+1)$
$=[x+(2y+3)][x+(y+1)]$
$=\boldsymbol{(x+2y+3)(x+y+1)}$

(6) $x^2-xy-2y^2-4x+11y-5$
$=x^2-(y+4)x-(2y^2-11y+5)$
$=x^2-(y+4)x-(2y-1)(y-5)$
$=[x-(2y-1)][x+(y-5)]$
$=\boldsymbol{(x-2y+1)(x+y-5)}$

(7) $x^2+xy-2y^2-x-11y-12$
$=x^2+(y-1)x-(2y^2+11y+12)$
$=x^2+(y-1)x-(2y+3)(y+4)$
$=[x+(2y+3)][x-(y+4)]$
$=\boldsymbol{(x+2y+3)(x-y-4)}$

J 35a KUMON

Factorisation III

Factorise the following expressions.

(1) $x^2+3xy+2y^2+4x+7y+3$
$=x^2+(3y+4)x+(2y^2+7y+3)$
$=x^2+(3y+4)x+(2y+1)(y+3)$
$=[x+(2y+1)][x+(y+3)]$
$=\boldsymbol{(x+2y+1)(x+y+3)}$

(2) $x^2-5xy+6y^2-3x+7y+2$
$=x^2-(5y+3)x+(6y^2+7y+2)$
$=x^2-(5y+3)x+(3y+2)(2y+1)$
$=[x-(3y+2)][x-(2y+1)]$
$=\boldsymbol{(x-3y-2)(x-2y-1)}$

(3) $x^2-3y^2+2xy+4x-8y-5$
$=x^2+2(y+2)x-(3y^2+8y+5)$
$=x^2+2(y+2)x-(3y+5)(y+1)$
$=[x+(3y+5)][x-(y+1)]$
$=\boldsymbol{(x+3y+5)(x-y-1)}$

(4) $x^2+6x+5-2y^2-xy-9y$
$=x^2-(y-6)x-(2y^2+9y-5)$
$=x^2-(y-6)x-(2y-1)(y+5)$
$=[x-(2y-1)][x+(y+5)]$
$=\boldsymbol{(x-2y+1)(x+y+5)}$

J35b

(5) $x^2-4y+3y^2+4xy-4$
$=x^2+4yx+(3y^2-4y-4)$
$=x^2+4yx+(3y+2)(y-2)$
$=[x+(3y+2)][x+(y-2)]$
$=\boldsymbol{(x+3y+2)(x+y-2)}$

(6) $x^2-xy-6y^2-x+13y-6$
$=x^2-(y+1)x-(6y^2-13y+6)$
$=x^2-(y+1)x-(3y-2)(2y-3)$
$=[x-(3y-2)][x+(2y-3)]$
$=\boldsymbol{(x-3y+2)(x+2y-3)}$

(7) $x^2+3y^2-4xy+4x-16y-12$
$=x^2-4(y-1)x+(3y^2-16y-12)$
$=x^2-4(y-1)x+(3y+2)(y-6)$
$=[x-(3y+2)][x-(y-6)]$
$=\boldsymbol{(x-3y-2)(x-y+6)}$

(8) $x^2-y-6y^2+xy-7x+12$
$=x^2+(y-7)x-(6y^2+y-12)$
$=x^2+(y-7)x-(3y-4)(2y+3)$
$=[x+(3y-4)][x-(2y+3)]$
$=\boldsymbol{(x+3y-4)(x-2y-3)}$

J 36a KUMON

Factorisation III

Factorise the following expressions.

Ex.

$$
\begin{aligned}
& x^2+4xy+4x+4y^2+8y+4 \\
= & \ x^2+4(y+1)x+4(y^2+2y+1) \\
= & \ x^2+4(y+1)x+4(y+1)^2 \\
= & \ [x+2(y+1)]^2 \\
= & \ (x+2y+2)^2
\end{aligned}
$$

Use the formula $a^2+2ab+b^2=(a+b)^2$.

(1) $x^2+2xy+2x+y^2+2y+1$
$=x^2+2(y+1)x+(y^2+2y+1)$
$=x^2+2(y+1)x+(y+1)^2$
$=[x+(y+1)]^2$
$=\boldsymbol{(x+y+1)^2}$

(2) $x^2-2xy-2x+y^2+2y+1$
$=x^2-2(y+1)x+(y^2+2y+1)$
$=x^2-2(y+1)x+(y+1)^2$
$=[x-(y+1)]^2$
$=\boldsymbol{(x-y-1)^2}$

(3) $x^2-4xy-4x+4y^2+8y+4$
$=x^2-4(y+1)x+4(y^2+2y+1)$
$=x^2-4(y+1)x+4(y+1)^2$
$=[x-2(y+1)]^2$
$=\boldsymbol{(x-2y-2)^2}$

(4) $x^2-6xy-6x+9y^2+18y+9$
$=x^2-6(y+1)x+9(y^2+2y+1)$
$=x^2-6(y+1)x+9(y+1)^2$
$=[x-3(y+1)]^2$
$=\boldsymbol{(x-3y-3)^2}$

J36b

Ex.

$$
\begin{aligned}
&x^2+y^2+z^2+2xy+2yz+2zx \\
&=x^2+2(y+z)x+(y^2+2yz+z^2) \\
&=x^2+2(y+z)x+(y+z)^2 \\
&=[x+(y+z)]^2 \\
&=(x+y+z)^2
\end{aligned}
$$

Arrange in powers of x.

(5) $x^2+y^2+9z^2+2xy+6xz+6yz$

$$
\begin{aligned}
&=x^2+2(y+3z)x+(y^2+6yz+9z^2) \\
&=x^2+2(y+3z)x+(y+3z)^2 \\
&=[x+(y+3z)]^2 \\
&=\boldsymbol{(x+y+3z)^2}
\end{aligned}
$$

(6) $x^2+4y^2+z^2+4xy+2xz+4yz$

$$
\begin{aligned}
&=x^2+2(2y+z)x+(4y^2+4yz+z^2) \\
&=x^2+2(2y+z)x+(2y+z)^2 \\
&=[x+(2y+z)]^2 \\
&=\boldsymbol{(x+2y+z)^2}
\end{aligned}
$$

(7) $x^2+4y^2+9z^2+4xy+6xz+12yz$

$$
\begin{aligned}
&=x^2+2(2y+3z)x+(4y^2+12yz+9z^2) \\
&=x^2+2(2y+3z)x+(2y+3z)^2 \\
&=[x+(2y+3z)]^2 \\
&=\boldsymbol{(x+2y+3z)^2}
\end{aligned}
$$

J 37a

KUMON

Factorisation III

Factorise the following expressions.

Ex.

$$\begin{aligned} & x^2+3xy+5x+2y^2+8y+6 \\ &= x^2+(3y+5)x+2(y^2+4y+3) \\ &= x^2+(3y+5)x+2(y+3)(y+1) \\ &= [x+2(y+1)][x+(y+3)] \\ &= (x+2y+2)(x+y+3) \end{aligned}$$

When multiplied: $2(y+3)(y+1)$
When added: $(3y+5)$
The terms are $2(y+1)$ and $(y+3)$.

(1) $x^2+3xy-3x+2y^2-8y-10$

$$\begin{aligned} &= x^2+3(y-1)x+2(y^2-4y-5) \\ &= x^2+3(y-1)x+2(y-5)(y+1) \\ &= [x+2(y+1)][x+(y-5)] \\ &= \boldsymbol{(x+2y+2)(x+y-5)} \end{aligned}$$

(2) $x^2+3xy-3x+2y^2-2y-4$

$$\begin{aligned} &= x^2+3(y-1)x+2(y^2-y-2) \\ &= x^2+3(y-1)x+2(y-2)(y+1) \\ &= [x+2(y-2)][x+(y+1)] \\ &= \boldsymbol{(x+2y-4)(x+y+1)} \end{aligned}$$

(3) $x^2-xy-x-2y^2+14y-20$

$$\begin{aligned} &= x^2-(y+1)x-2(y^2-7y+10) \\ &= x^2-(y+1)x-2(y-5)(y-2) \\ &= [x-2(y-2)][x+(y-5)] \\ &= \boldsymbol{(x-2y+4)(x+y-5)} \end{aligned}$$

J 37b

Ex.

$$
\begin{aligned}
&x^2+4xy+4x+3y^2+6y+3\\
&=x^2+4(y+1)x+3(y^2+2y+1)\\
&=x^2+4(y+1)x+3(y+1)^2\\
&=[x+3(y+1)][x+(y+1)]\\
&=(x+3y+3)(x+y+1)
\end{aligned}
$$

When multiplied: $3(y+1)^2$
When added: $4(y+1)$
The terms are $3(y+1)$ and $(y+1)$.

(4) $x^2+5xy+5x+6y^2+12y+6$
$=x^2+5(y+1)x+6(y^2+2y+1)$
$=x^2+5(y+1)x+6(y+1)^2$
$=[x+3(y+1)][x+2(y+1)]$
$=\boldsymbol{(x+3y+3)(x+2y+2)}$

(5) $x^2+5xy+5x-6y^2-12y-6$
$=x^2+5(y+1)x-6(y^2+2y+1)$
$=x^2+5(y+1)x-6(y+1)^2$
$=[x-(y+1)][x+6(y+1)]$
$=\boldsymbol{(x-y-1)(x+6y+6)}$

(6) $x^2+4xy+4x-12y^2-24y-12$
$=x^2+4(y+1)x-12(y^2+2y+1)$
$=x^2+4(y+1)x-12(y+1)^2$
$=[x-2(y+1)][x+6(y+1)]$
$=\boldsymbol{(x-2y-2)(x+6y+6)}$

J 38a

Factorisation III

Factorise the following expressions.

Ex.

$$2x^2+(a+6b)x+3ab$$
$$=(2x+a)(x+3b)$$

Factorise into $(2x+\triangle)(x+\square)$.

Two possibilities are:

$$\begin{array}{ccccc} 2 & & a & \rightarrow & a \\ & \times & & & \\ 1 & & 3b & \rightarrow & \underline{6b} \\ & & & & a+6b \end{array} \qquad \begin{array}{ccccc} 2 & & 3b & \rightarrow & 3b \\ & \times & & & \\ 1 & & a & \rightarrow & \underline{2a} \\ & & & & 2a+3b \end{array}$$

The first is correct.

(1) $2x^2-(a+6b)x+3ab$

$=\boldsymbol{(2x-a)(x-3b)}$

$$\begin{array}{ccccc} 2 & & -a & \rightarrow & -a \\ & \times & & & \\ 1 & & -3b & \rightarrow & \underline{-6b} \\ & & & & -(a+6b) \end{array}$$

(2) $2x^2+(a-6b)x-3ab$

$=\boldsymbol{(2x+a)(x-3b)}$

$$\begin{array}{ccccc} 2 & & a & \rightarrow & a \\ & \times & & & \\ 1 & & -3b & \rightarrow & \underline{-6b} \\ & & & & a-6b \end{array}$$

(3) $2x^2+(2a+3b)x+3ab$

$=\boldsymbol{(2x+3b)(x+a)}$

$$\begin{array}{ccccc} 2 & & 3b & \rightarrow & 3b \\ & \times & & & \\ 1 & & a & \rightarrow & \underline{2a} \\ & & & & 2a+3b \end{array}$$

Further notes on question (1).

— Looking at $2x^2$, write the factors of 2 on the left.

$$\begin{array}{cc} 2 & \\ & \times \\ 1 & \end{array}$$

— Looking at $3ab$, write the factors of $3ab$ on the right.

$$\begin{array}{ccc} 2 & & -a \\ & \times & \\ 1 & & -3b \end{array}$$

— Looking at $-(a+6b)x$, check these terms multiply to give $-(a+6b)$.

$$\begin{array}{ccccc} 2 & & -a & \rightarrow & -a \\ & \times & & & \\ 1 & & -3b & \rightarrow & \underline{-6b} \\ & & & & -(a+6b) \end{array}$$

— It is often necessary to try different combinations of factors and signs.

— Read off the answer directly.

$$\begin{array}{ccl} 2 & & -a \text{ gives } (2x-a) \\ & \times & \\ 1 & & -3b \text{ gives } (x-3b) \end{array}$$

J38b

(4) $2x^2+(2a-3b)x-3ab$

$=\boldsymbol{(2x-3b)(x+a)}$

$$\left[\begin{array}{ccccc} 2 & \diagdown\!\!\!\!\diagup & -3b & \to & -3b \\ 1 & & a & \to & \underline{2a} \\ & & & & 2a-3b \end{array}\right]$$

(5) $2x^2-(2a-3b)x-3ab$

$=\boldsymbol{(2x+3b)(x-a)}$

$$\left[\begin{array}{ccccc} 2 & \diagdown\!\!\!\!\diagup & 3b & \to & 3b \\ 1 & & -a & \to & \underline{-2a} \\ & & & & -(2a-3b) \end{array}\right]$$

(6) $2x^2+(3a+8)x+a(a+4)$

$=\boldsymbol{(2x+a)(x+a+4)}$

$$\left[\begin{array}{ccccc} 2 & \diagdown\!\!\!\!\diagup & a & \to & a \\ 1 & & a+4 & \to & \underline{2a+8} \\ & & & & 3a+8 \end{array}\right]$$

(7) $2x^2+(a+6)x-a(a+3)$

$=\boldsymbol{(2x-a)(x+a+3)}$

$$\left[\begin{array}{ccccc} 2 & \diagdown\!\!\!\!\diagup & -a & \to & -a \\ 1 & & a+3 & \to & \underline{2a+6} \\ & & & & a+6 \end{array}\right]$$

(8) $2x^2+(3a+4b)x+a(a+4b)$

$=\boldsymbol{(2x+a+4b)(x+a)}$

$$\left[\begin{array}{ccccc} 2 & \diagdown\!\!\!\!\diagup & a+4b & \to & a+4b \\ 1 & & a & \to & \underline{2a} \\ & & & & 3a+4b \end{array}\right]$$

J 39a KUMON

Factorisation III

Factorise the following expressions.

Ex.

$$
\begin{aligned}
&2x^2+5xy+3y^2+6x+7y+4\\
&=2x^2+(5y+6)x+(3y^2+7y+4)\\
&=2x^2+(5y+6)x+(3y+4)(y+1)\\
&=[2x+(3y+4)][x+(y+1)]\\
&=(2x+3y+4)(x+y+1)
\end{aligned}
$$

Note:
$2\times(y+1)+1\times(3y+4)=5y+6$

(1) $2x^2+7xy+3y^2+9x+7y+4$

$$
\begin{aligned}
&=2x^2+(7y+9)x+(3y^2+7y+4)\\
&=2x^2+(7y+9)x+(3y+4)(y+1)\\
&=[2x+(y+1)][x+(3y+4)]\\
&=\mathbf{(2x+y+1)(x+3y+4)}
\end{aligned}
$$

$$\left[\begin{array}{llcl} 2 & y+1 & \to & y+1 \\ 1 \times & 3y+4 & \to & \underline{6y+8} \\ & & & 7y+9 \end{array}\right]$$

(2) $2x^2+7xy+3y^2-9x-7y+4$

$$
\begin{aligned}
&=2x^2+(7y-9)x+(3y^2-7y+4)\\
&=2x^2+(7y-9)x+(3y-4)(y-1)\\
&=[2x+(y-1)][x+(3y-4)]\\
&=\mathbf{(2x+y-1)(x+3y-4)}
\end{aligned}
$$

$$\left[\begin{array}{llcl} 2 & y-1 & \to & y-1 \\ 1 \times & 3y-4 & \to & \underline{6y-8} \\ & & & 7y-9 \end{array}\right]$$

(3) $2x^2+8xy+6y^2+11x+13y+5$

$$
\begin{aligned}
&=2x^2+(8y+11)x+(6y^2+13y+5)\\
&=2x^2+(8y+11)x+(3y+5)(2y+1)\\
&=[2x+(2y+1)][x+(3y+5)]\\
&=\mathbf{(2x+2y+1)(x+3y+5)}
\end{aligned}
$$

$$\left[\begin{array}{llcl} 2 & 2y+1 & \to & 2y+1 \\ 1 \times & 3y+5 & \to & \underline{6y+10} \\ & & & 8y+11 \end{array}\right]$$

J 39b

(4) $3x^2+7xy+2y^2+11x+7y+6$

$$
\begin{aligned}
&= 3x^2+(7y+11)x+(2y^2+7y+6)\\
&= 3x^2+(7y+11)x+(2y+3)(y+2)\\
&= [3x+(y+2)][x+(2y+3)]\\
&= \boldsymbol{(3x+y+2)(x+2y+3)}
\end{aligned}
$$

$$
\left[\begin{array}{cclcl}
3 & \diagdown\!\!\!\!\diagup & y+2 & \to & y+2 \\
1 & & 2y+3 & \to & \underline{6y+9} \\
 & & & & 7y+11
\end{array}\right]
$$

(5) $2x^2-7xy+3y^2-9x+7y+4$

$$
\begin{aligned}
&= 2x^2-(7y+9)x+(3y^2+7y+4)\\
&= 2x^2-(7y+9)x+(3y+4)(y+1)\\
&= [2x-(y+1)][x-(3y+4)]\\
&= \boldsymbol{(2x-y-1)(x-3y-4)}
\end{aligned}
$$

$$
\left[\begin{array}{cclcl}
2 & \diagdown\!\!\!\!\diagup & -(y+1) & \to & -y-1 \\
1 & & -(3y+4) & \to & \underline{-6y-8} \\
 & & & & -(7y+9)
\end{array}\right]
$$

(6) $2x^2-8xy+6y^2-11x+13y+5$

$$
\begin{aligned}
&= 2x^2-(8y+11)x+(6y^2+13y+5)\\
&= 2x^2-(8y+11)x+(3y+5)(2y+1)\\
&= [2x-(2y+1)][x-(3y+5)]\\
&= \boldsymbol{(2x-2y-1)(x-3y-5)}
\end{aligned}
$$

$$
\left[\begin{array}{cclcl}
2 & \diagdown\!\!\!\!\diagup & -(2y+1) & \to & -2y-1 \\
1 & & -(3y+5) & \to & \underline{-6y-10} \\
 & & & & -(8y+11)
\end{array}\right]
$$

(7) $3x^2-7xy+2y^2-11x+7y+6$

$$
\begin{aligned}
&= 3x^2-(7y+11)x+(2y^2+7y+6)\\
&= 3x^2-(7y+11)x+(2y+3)(y+2)\\
&= [3x-(y+2)][x-(2y+3)]\\
&= \boldsymbol{(3x-y-2)(x-2y-3)}
\end{aligned}
$$

$$
\left[\begin{array}{cclcl}
3 & \diagdown\!\!\!\!\diagup & -(y+2) & \to & -y-2 \\
1 & & -(2y+3) & \to & \underline{-6y-9} \\
 & & & & -(7y+11)
\end{array}\right]
$$

J 40a KUMON

Factorisation III

Factorise the following expressions.

(1) $x^2+3(y+z)x+(y+2z)(2y+z)$

$=[x+(y+2z)][x+(2y+z)]$

$=\boldsymbol{(x+y+2z)(x+2y+z)}$

(2) $x^2+(y-z)x-(y+2z)(2y+z)$

$=[x-(y+2z)][x+(2y+z)]$

$=\boldsymbol{(x-y-2z)(x+2y+z)}$

(3) $x^2-(y-z)x-(y+2z)(2y+z)$

$=[x+(y+2z)][x-(2y+z)]$

$=\boldsymbol{(x+y+2z)(x-2y-z)}$

(4) $a^2-2b^2-3c^2-ab-2ac-5bc$

$=a^2-(b+2c)a-(2b^2+5bc+3c^2)$

$=a^2-(b+2c)a-(2b+3c)(b+c)$

$=[a-(2b+3c)][a+(b+c)]$

$=\boldsymbol{(a-2b-3c)(a+b+c)}$

J40b

(5)　$a^2-8b^2+2c^2+2ab+3ac$

$=a^2+(2b+3c)a-2(4b^2-c^2)$

$=a^2+(2b+3c)a-2(2b+c)(2b-c)$

$=[a+2(2b+c)][a-(2b-c)]$

$=\boldsymbol{(a+4b+2c)(a-2b+c)}$

(6)　$x^2-4y^2+3x-2y+2$

$=x^2+3x-2(2y^2+y-1)$

$=x^2+3x-2(2y-1)(y+1)$

$=[x-(2y-1)][x+2(y+1)]$

$=\boldsymbol{(x-2y+1)(x+2y+2)}$

(7)　$2x^2-(5a-4b)x-(a+2b)(3a-b)$

$=[2x+(a+2b)][x-(3a-b)]$

$=\boldsymbol{(2x+a+2b)(x-3a+b)}$

$$\left[\begin{array}{llll} 2 & a+2b & \rightarrow & a+2b \\ 1 & -(3a-b) & \rightarrow & \underline{-6a+2b} \\ & & & -(5a-4b) \end{array}\right]$$

(8)　$2x^2-7xy+3y^2+9x-7y+4$

$=2x^2-(7y-9)x+(3y^2-7y+4)$

$=2x^2-(7y-9)x+(3y-4)(y-1)$

$=[2x-(y-1)][x-(3y-4)]$

$=\boldsymbol{(2x-y+1)(x-3y+4)}$

$$\left[\begin{array}{llll} 2 & -(y-1) & \rightarrow & -y+1 \\ 1 & -(3y-4) & \rightarrow & \underline{-6y+8} \\ & & & -(7y-9) \end{array}\right]$$

J41a KUMON

Factorisation IV

Factorise the following expressions.

(1) $2x^2-3xy-2y^2-2x-11y-12$

$=2x^2-(3y+2)x-(2y^2+11y+12)$

$=2x^2-(3y+2)x-(2y+3)(y+4)$

$=[2x+(y+4)][x-(2y+3)]$

$=\mathbf{(2x+y+4)(x-2y-3)}$

$$\left[\begin{array}{cclcl} 2 & \times & y+4 & \to & y+4 \\ 1 & & -(2y+3) & \to & \underline{-4y-6} \\ & & & & -(3y+2) \end{array}\right]$$

(2) $2x^2+3xy-2y^2+2x-11y-12$

$=2x^2+(3y+2)x-(2y^2+11y+12)$

$=2x^2+(3y+2)x-(2y+3)(y+4)$

$=[2x-(y+4)][x+(2y+3)]$

$=\mathbf{(2x-y-4)(x+2y+3)}$

$$\left[\begin{array}{cclcl} 2 & \times & -(y+4) & \to & -y-4 \\ 1 & & 2y+3 & \to & \underline{4y+6} \\ & & & & 3y+2 \end{array}\right]$$

(3) $2x^2+7xy+3y^2+13x+14y+15$

$=2x^2+(7y+13)x+(3y^2+14y+15)$

$=2x^2+(7y+13)x+(3y+5)(y+3)$

$=[2x+(y+3)][x+(3y+5)]$

$=\mathbf{(2x+y+3)(x+3y+5)}$

$$\left[\begin{array}{cclcl} 2 & \times & y+3 & \to & y+3 \\ 1 & & 3y+5 & \to & \underline{6y+10} \\ & & & & 7y+13 \end{array}\right]$$

J41b

(4) $2x^2-7xy+6y^2+7x-11y+3$
$=2x^2-7(y-1)x+(6y^2-11y+3)$
$=2x^2-7(y-1)x+(3y-1)(2y-3)$
$=[2x-(3y-1)][x-(2y-3)]$
$=\boldsymbol{(2x-3y+1)(x-2y+3)}$

$$\left[\begin{array}{llcl} 2 & -(3y-1) & \to & -3y+1 \\ 1 & -(2y-3) & \to & \underline{-4y+6} \\ & & & -7(y-1) \end{array}\right]$$

(5) $2x^2+xy-y^2+3x-3y-2$
$=2x^2+(y+3)x-(y^2+3y+2)$
$=2x^2+(y+3)x-(y+2)(y+1)$
$=[2x-(y+1)][x+(y+2)]$
$=\boldsymbol{(2x-y-1)(x+y+2)}$

$$\left[\begin{array}{llcl} 2 & -(y+1) & \to & -y-1 \\ 1 & y+2 & \to & \underline{2y+4} \\ & & & y+3 \end{array}\right]$$

(6) $2x^2+xy-y^2+3x+1$
$=2x^2+(y+3)x-(y^2-1)$
$=2x^2+(y+3)x-(y+1)(y-1)$
$=[2x-(y-1)][x+(y+1)]$
$=\boldsymbol{(2x-y+1)(x+y+1)}$

$$\left[\begin{array}{llcl} 2 & -(y-1) & \to & -y+1 \\ 1 & y+1 & \to & \underline{2y+2} \\ & & & y+3 \end{array}\right]$$

(7) $2x^2-5xy-3y^2-14y-8$
$=2x^2-5yx-(3y^2+14y+8)$
$=2x^2-5yx-(3y+2)(y+4)$
$=[2x+(y+4)][x-(3y+2)]$
$=\boldsymbol{(2x+y+4)(x-3y-2)}$

$$\left[\begin{array}{llcl} 2 & y+4 & \to & y+4 \\ 1 & -(3y+2) & \to & \underline{-6y-4} \\ & & & -5y \end{array}\right]$$

J 42a KUMON

Factorisation IV

Arrange in descending powers of a, then factorise.

Ex.

$$
\begin{aligned}
&2a^2+2b^2+c^2+4ab+3ac+3bc\\
&=2a^2+(4b+3c)a+(2b^2+3bc+c^2)\\
&=2a^2+(4b+3c)a+(2b+c)(b+c)\\
&=[2a+(2b+c)][a+(b+c)]\\
&=(2a+2b+c)(a+b+c)
\end{aligned}
$$

$$
\left[\begin{array}{cccc}
2 & \times & 2b+c & \to\ 2b+c \\
1 & & b+c & \to\ \underline{2b+2c} \\
 & & & 4b+3c
\end{array}\right]
$$

(1) $2a^2+b^2+2c^2+3ab+5ac+3bc$

$$
\begin{aligned}
&=2a^2+(3b+5c)a+(b^2+3bc+2c^2)\\
&=2a^2+(3b+5c)a+(b+2c)(b+c)\\
&=[2a+(b+c)][a+(b+2c)]\\
&=\boldsymbol{(2a+b+c)(a+b+2c)}
\end{aligned}
$$

$$
\left[\begin{array}{cccc}
2 & \times & b+c & \to\ b+c \\
1 & & b+2c & \to\ \underline{2b+4c} \\
 & & & 3b+5c
\end{array}\right]
$$

(2) $2a^2+2b^2+2c^2-4ab-5ac+5bc$

$$
\begin{aligned}
&=2a^2-(4b+5c)a+(2b^2+5bc+2c^2)\\
&=2a^2-(4b+5c)a+(2b+c)(b+2c)\\
&=[2a-(2b+c)][a-(b+2c)]\\
&=\boldsymbol{(2a-2b-c)(a-b-2c)}
\end{aligned}
$$

$$
\left[\begin{array}{cccc}
2 & \times & -(2b+c) & \to\ -2b-c \\
1 & & -(b+2c) & \to\ \underline{-2b-4c} \\
 & & & -(4b+5c)
\end{array}\right]
$$

J42b

(3) $2a^2-3b^2-4c^2+5ab-2ac+8bc$

$$
\begin{aligned}
&=2a^2+(5b-2c)a-(3b^2-8bc+4c^2)\\
&=2a^2+(5b-2c)a-(3b-2c)(b-2c)\\
&=[2a-(b-2c)][a+(3b-2c)]\\
&=\boldsymbol{(2a-b+2c)(a+3b-2c)}
\end{aligned}
$$

$$
\left[\begin{array}{llll}
2 & -(b-2c) & \to & -b+2c \\
1 & 3b-2c & \to & \underline{6b-4c} \\
 & & & 5b-2c
\end{array}\right]
$$

(4) $3a^2-6b^2-2c^2-7ab+5ac+7bc$

$$
\begin{aligned}
&=3a^2-(7b-5c)a-(6b^2-7bc+2c^2)\\
&=3a^2-(7b-5c)a-(3b-2c)(2b-c)\\
&=[3a+(2b-c)][a-(3b-2c)]\\
&=\boldsymbol{(3a+2b-c)(a-3b+2c)}
\end{aligned}
$$

$$
\left[\begin{array}{llll}
3 & 2b-c & \to & 2b-c \\
1 & -(3b-2c) & \to & \underline{-9b+6c} \\
 & & & -(7b-5c)
\end{array}\right]
$$

(5) $2a^2-2b^2-3c^2+3ab+5ac-5bc$

$$
\begin{aligned}
&=2a^2+(3b+5c)a-(2b^2+5bc+3c^2)\\
&=2a^2+(3b+5c)a-(2b+3c)(b+c)\\
&=[2a-(b+c)][a+(2b+3c)]\\
&=\boldsymbol{(2a-b-c)(a+2b+3c)}
\end{aligned}
$$

$$
\left[\begin{array}{llll}
2 & -(b+c) & \to & -b-c \\
1 & 2b+3c & \to & \underline{4b+6c} \\
 & & & 3b+5c
\end{array}\right]
$$

(6) $2a^2-2b^2-c^2-3ab+ac+3bc$

$$
\begin{aligned}
&=2a^2-(3b-c)a-(2b^2-3bc+c^2)\\
&=2a^2-(3b-c)a-(2b-c)(b-c)\\
&=[2a+(b-c)][a-(2b-c)]\\
&=\boldsymbol{(2a+b-c)(a-2b+c)}
\end{aligned}
$$

$$
\left[\begin{array}{llll}
2 & b-c & \to & b-c \\
1 & -(2b-c) & \to & \underline{-4b+2c} \\
 & & & -(3b-c)
\end{array}\right]
$$

J43a KUMON

Factorisation IV

1. Arrange in descending powers of b, then factorise.

Ex.

$$\begin{aligned}&2a^2+2b^2+c^2+4ab+3ac+3bc\\=&2b^2+(4a+3c)b+(2a^2+3ac+c^2)\\=&2b^2+(4a+3c)b+(2a+c)(a+c)\\=&[2b+(2a+c)][b+(a+c)]\\=&(2b+2a+c)(b+a+c)\end{aligned}$$

(1) $3a^2+2b^2+6c^2+7ab+11ac+7bc$

$=2b^2+7(a+c)b+(3a^2+11ac+6c^2)$

$=2b^2+7(a+c)b+(3a+2c)(a+3c)$

$=[2b+(a+3c)][b+(3a+2c)]$

$=\boldsymbol{(2b+a+3c)(b+3a+2c)}$

(2) $2a^2+2b^2+2c^2-4ab-5ac+5bc$

$=2b^2-(4a-5c)b+(2a^2-5ac+2c^2)$

$=2b^2-(4a-5c)b+(2a-c)(a-2c)$

$=[2b-(2a-c)][b-(a-2c)]$

$=\boldsymbol{(2b-2a+c)(b-a+2c)}$

Note: Looking at the example on J42a and the example above, we can see that we get the same result by arranging in powers of a, b or c.

J43b

2. Consider $3a^2-5ab+2b^2-a-b-10$.
Arrange (1) in powers of a,
(2) in powers of b,
then factorise.

(1) In powers of a,

$$\begin{aligned}
&3a^2-5ab+2b^2-a-b-10\\
=&3a^2-(5b+1)a+(2b^2-b-10)\\
=&3a^2-(5b+1)a+(2b-5)(b+2)\\
=&[3a-(2b-5)][a-(b+2)]\\
=&\boldsymbol{(3a-2b+5)(a-b-2)}
\end{aligned}$$

(2) In powers of b,

$$\begin{aligned}
&3a^2-5ab+2b^2-a-b-10\\
=&2b^2-(5a+1)b+(3a^2-a-10)\\
=&2b^2-(5a+1)b+(3a+5)(a-2)\\
=&[2b-(3a+5)][b-(a-2)]\\
=&\boldsymbol{(2b-3a-5)(b-a+2)}
\end{aligned}$$

J 44 a KUMON

Factorisation IV

1. Factorise using the difference of two squares.
(i.e. rewrite as two terms A^2-B^2)

Ex.

$$\begin{aligned}
&4a^2+4ab+b^2-c^2\\
&=(4a^2+4ab+b^2)-c^2\\
&=(2a+b)^2-c^2\\
&=[(2a+b)+c][(2a+b)-c]\\
&=(2a+b+c)(2a+b-c)
\end{aligned}$$

(1) $a^2+6ab+9b^2-c^2$
$=(a^2+6ab+9b^2)-c^2$
$=(a+3b)^2-c^2$
$=[(a+3b)+c][(a+3b)-c]$
$=\boldsymbol{(a+3b+c)(a+3b-c)}$

(2) $4a^2-12ab+9b^2-4c^2$
$=(4a^2-12ab+9b^2)-4c^2$
$=(2a-3b)^2-4c^2$
$=[(2a-3b)+2c][(2a-3b)-2c]$
$=\boldsymbol{(2a-3b+2c)(2a-3b-2c)}$

(3) $4a^2-b^2+6bc-9c^2$
$=4a^2-(b^2-6bc+9c^2)$
$=4a^2-(b-3c)^2$
$=[2a+(b-3c)][2a-(b-3c)]$
$=\boldsymbol{(2a+b-3c)(2a-b+3c)}$

J44b

2. Consider $a^2+2ab+b^2-x^2-6x-9$.

Arrange (1) in powers of a,
(2) in powers of b,
(3) as the difference of two squares,
then factorise.

(1) In powers of a,

$$
\begin{aligned}
& a^2+2ab+b^2-x^2-6x-9 \\
&= a^2+2ba+[b^2-(x^2+6x+9)] \\
&= a^2+2ba+[b^2-(x+3)^2] \\
&= a^2+2ba+(b+x+3)(b-x-3) \\
&= [a+(b+x+3)][a+(b-x-3)] \\
&= \boldsymbol{(a+b+x+3)(a+b-x-3)}
\end{aligned}
$$

(2) In powers of b,

$$
\begin{aligned}
& a^2+2ab+b^2-x^2-6x-9 \\
&= b^2+2ab+[a^2-(x^2+6x+9)] \\
&= b^2+2ab+[a^2-(x+3)^2] \\
&= b^2+2ab+(a+x+3)(a-x-3) \\
&= [b+(a+x+3)][b+(a-x-3)] \\
&= \boldsymbol{(b+a+x+3)(b+a-x-3)}
\end{aligned}
$$

(3) As the difference of two squares,

$$
\begin{aligned}
& a^2+2ab+b^2-x^2-6x-9 \\
&= (a^2+2ab+b^2)-(x^2+6x+9) \\
&= (a+b)^2-(x+3)^2 \\
&= [(a+b)+(x+3)][(a+b)-(x+3)] \\
&= \boldsymbol{(a+b+x+3)(a+b-x-3)}
\end{aligned}
$$

J 45a KUMON

Factorisation IV

Factorise by any method.

(1) x^2-y^2-4x+4
$=(x^2-4x+4)-y^2$
$=(x-2)^2-y^2$
$=[(x-2)+y][(x-2)-y]$
$=\boldsymbol{(x+y-2)(x-y-2)}$

(2) x^2-y^2-6y-9
$=x^2-(y^2+6y+9)$
$=x^2-(y+3)^2$
$=[x+(y+3)][x-(y+3)]$
$=\boldsymbol{(x+y+3)(x-y-3)}$

(3) $4x^2-12xy+9y^2-9$
$=(4x^2-12xy+9y^2)-9$
$=(2x-3y)^2-9$
$=[(2x-3y)+3][(2x-3y)-3]$
$=\boldsymbol{(2x-3y+3)(2x-3y-3)}$

J45b

（4） $x^2-y^2+4ay-4a^2$

$=x^2-(y^2-4ay+4a^2)$

$=x^2-(y-2a)^2$

$=[x+(y-2a)][x-(y-2a)]$

$=\boldsymbol{(x+y-2a)(x-y+2a)}$

（5） $x^2-6ax-y^2+9a^2$

$=(x^2-6ax+9a^2)-y^2$

$=(x-3a)^2-y^2$

$=[(x-3a)+y][(x-3a)-y]$

$=\boldsymbol{(x-3a+y)(x-3a-y)}$

（6） $9x^2+4yz-4y^2-z^2$

$=9x^2-(4y^2-4yz+z^2)$

$=9x^2-(2y-z)^2$

$=[3x+(2y-z)][3x-(2y-z)]$

$=\boldsymbol{(3x+2y-z)(3x-2y+z)}$

（7） $x^2+4y^2-z^2-4+4xy-4z$

$=(x^2+4xy+4y^2)-(z^2+4z+4)$

$=(x+2y)^2-(z+2)^2$

$=[(x+2y)+(z+2)][(x+2y)-(z+2)]$

$=\boldsymbol{(x+2y+z+2)(x+2y-z-2)}$

J46a KUMON

Factorisation IV

Factorise by any method.

(1) $a^2-b^2-c^2+d^2+2ad+2bc$
$=(a^2+2ad+d^2)-(b^2-2bc+c^2)$
$=(a+d)^2-(b-c)^2$
$=[(a+d)+(b-c)][(a+d)-(b-c)]$
$=\boldsymbol{(a+d+b-c)(a+d-b+c)}$

(2) $2ad-2bc-a^2+b^2+c^2-d^2$
$=(b^2-2bc+c^2)-(a^2-2ad+d^2)$
$=(b-c)^2-(a-d)^2$
$=[(b-c)+(a-d)][(b-c)-(a-d)]$
$=\boldsymbol{(b-c+a-d)(b-c-a+d)}$

(3) $a^2+b^2+2bc-2ca-2ab$
$=(a^2-2ab+b^2)+2bc-2ca$
$=(a-b)^2+2c(b-a)$
$=(a-b)^2-2c(a-b)$
$=(a-b)[(a-b)-2c]$
$=\boldsymbol{(a-b)(a-b-2c)}$

J46b

(4) $ax^2-bx^2-2bx+2ax-3a+3b$

$=(a-b)x^2+2(a-b)x-3(a-b)$

$=(a-b)(x^2+2x-3)$

$=\boldsymbol{(a-b)(x+3)(x-1)}$

(5) $2x^2+xy-y^2+3x-3y-2$

$=2x^2+(y+3)x-(y^2+3y+2)$

$=2x^2+(y+3)x-(y+2)(y+1)$

$=[2x-(y+1)][x+(y+2)]$

$=\boldsymbol{(2x-y-1)(x+y+2)}$

(6) $2x^2+xy-y^2+3x+1$

$=2x^2+(y+3)x-(y^2-1)$

$=2x^2+(y+3)x-(y+1)(y-1)$

$=[2x-(y-1)][x+(y+1)]$

$=\boldsymbol{(2x-y+1)(x+y+1)}$

(7) $x^2-4y^2-x+6y-2$

$=x^2-x-2(2y^2-3y+1)$

$=x^2-x-2(2y-1)(y-1)$

$=[x-(2y-1)][x+2(y-1)]$

$=\boldsymbol{(x-2y+1)(x+2y-2)}$

J 47a KUMON

Factorisation IV

Formula

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

These formulas can be called the ***sum of two cubes*** and the ***difference of two cubes***.

Factorise the following expressions.

(1) $x^3+8y^3=\boldsymbol{(x+2y)(x^2-2xy+4y^2)}$

(2) $8x^3-27y^3=\boldsymbol{(2x-3y)(4x^2+6xy+9y^2)}$

(3) $a^3-8b^3=\boldsymbol{(a-2b)(a^2+2ab+4b^2)}$

(4) $8a^3+27b^3=\boldsymbol{(2a+3b)(4a^2-6ab+9b^2)}$

(5) $a^3-64=\boldsymbol{(a-4)(a^2+4a+16)}$

J47b

(6) $27a^3-8b^3=\boldsymbol{(3a-2b)(9a^2+6ab+4b^2)}$

(7) $1-x^3=\boldsymbol{(1-x)(1+x+x^2)}$

(8) $a^3+8b^6=\boldsymbol{(a+2b^2)(a^2-2ab^2+4b^4)}$

(9) $a^3-27b^9=\boldsymbol{(a-3b^3)(a^2+3ab^3+9b^6)}$

(10) $x^6+y^9=\boldsymbol{(x^2+y^3)(x^4-x^2y^3+y^6)}$

J 48a

KUMON

Factorisation IV

Formula

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

Factorise the following expressions.

(1) $125a^3+8b^3=\mathbf{(5a+2b)(25a^2-10ab+4b^2)}$

(2) $27x^3-64=\mathbf{(3x-4)(9x^2+12x+16)}$

(3) $2x^3+16=2(\ \mathbf{x^3+8}\ )=\mathbf{2(x+2)(x^2-2x+4)}$

(4) $64a^4-27a=a(64a^3-27)=\mathbf{a(4a-3)(16a^2+12a+9)}$

(5) $64x-x^4=x(64-x^3)=\mathbf{x(4-x)(16+4x+x^2)}$

J48b

(6) $(a+b)^3-8b^3$

$=[(a+b)-2b][(a+b)^2+(a+b)\cdot(2b)+(2b)^2]$

$=(a-b)(a^2+2ab+b^2+2ab+2b^2+4b^2)$

$=\boldsymbol{(a-b)(a^2+4ab+7b^2)}$

(7) $8(x+y)^3-y^3$

$=[2(x+y)-y][4(x+y)^2+2(x+y)y+y^2]$

$=(2x+2y-y)(4x^2+8xy+4y^2+2xy+2y^2+y^2)$

$=\boldsymbol{(2x+y)(4x^2+10xy+7y^2)}$

(8) $(a+b)^3-b^3$

$=[(a+b)-b][(a+b)^2+(a+b)b+b^2]$

$=a(a^2+2ab+b^2+ab+b^2+b^2)$

$=\boldsymbol{a(a^2+3ab+3b^2)}$

(9) $(a+b)^3-(b-c)^3$

$=[(a+b)-(b-c)][(a+b)^2+(a+b)(b-c)+(b-c)^2]$

$=(a+c)(a^2+2ab+b^2+ab-ac+b^2-bc+b^2-2bc+c^2)$

$=\boldsymbol{(a+c)(a^2+3b^2+c^2+3ab-3bc-ac)}$

J49a KUMON

Factorisation IV

Factorise the following expressions.

(1) a^6-b^6

$a^6-b^6=(a^2)^3-(b^2)^3$

$=(a^2-b^2)(a^4+a^2b^2+b^4)$

$=(a+\boxed{\boldsymbol{b}})(a-\boxed{\boldsymbol{b}})[(a^2+b^2)^2-a^2b^2]$

$=(\ \boldsymbol{a+b}\ )(\ \boldsymbol{a-b}\ )[(a^2+b^2)+ab][(\ \boldsymbol{a^2+b^2}\ )-ab]$

$=\boldsymbol{(a+b)(a-b)(a^2+b^2+ab)(a^2+b^2-ab)}$

(2) a^6-b^6

$a^6-b^6=(a^3)^2-(b^3)^2$

$=(a^3+\boxed{\boldsymbol{b^3}})(a^3-\boxed{\boldsymbol{b^3}})$

$=\boldsymbol{(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)}$

(3) a^6+b^6

$=\boldsymbol{(a^2+b^2)(a^4-a^2b^2+b^4)}$

(4) a^9-b^9

$=(a^3-b^3)(a^6+a^3b^3+b^6)$

$=\boldsymbol{(a-b)(a^2+ab+b^2)(a^6+a^3b^3+b^6)}$

(5) $a^{12}+b^{12}$

$=\boldsymbol{(a^4+b^4)(a^8-a^4b^4+b^8)}$

Note: In (1) and (2), both methods give the same answer.

J49b

Ex.

$$
\begin{aligned}
&(x^3+y^3)+xy(x+y)\\
&=(x+y)(x^2-xy+y^2)+xy(x+y)\\
&=(x+y)(x^2+y^2)
\end{aligned}
$$

(6) $xy(x-y)+x^3-y^3$

$$
\begin{aligned}
&=xy(x-y)+(x-y)(x^2+xy+y^2)\\
&=(x-y)(x^2+2xy+y^2)\\
&=\boldsymbol{(x-y)(x+y)^2}
\end{aligned}
$$

(7) a^3+b^3+a+b

$$
\begin{aligned}
&=(a+b)(a^2-ab+b^2)+(a+b)\\
&=\boldsymbol{(a+b)(a^2-ab+b^2+1)}
\end{aligned}
$$

(8) $a^3-b^3-3ab(a-b)$

$$
\begin{aligned}
&=(a-b)(a^2+ab+b^2)-3ab(a-b)\\
&=(a-b)(a^2+ab+b^2-3ab)\\
&=(a-b)(a^2-2ab+b^2)\\
&=(a-b)(a-b)^2\\
&=\boldsymbol{(a-b)^3}
\end{aligned}
$$

(9) $4-x^2+4x^3-x^5$

$$
\begin{aligned}
&=(4-x^2)+x^3(4-x^2)\\
&=(4-x^2)(1+x^3)\\
&=\boldsymbol{(2+x)(2-x)(1+x)(1-x+x^2)}
\end{aligned}
$$

J 50a

KUMON

Factorisation IV

Factorise the following expressions.

(1) $2x^2-7xy+3y^2+9x-7y+4$
$=2x^2-(7y-9)x+(3y^2-7y+4)$
$=2x^2-(7y-9)x+(3y-4)(y-1)$
$=[2x-(y-1)][x-(3y-4)]$
$=\mathbf{(2x-y+1)(x-3y+4)}$

(2) $3x^2+7xy+2y^2+11x+7y+6$
$=3x^2+(7y+11)x+(2y^2+7y+6)$
$=3x^2+(7y+11)x+(2y+3)(y+2)$
$=[3x+(y+2)][x+(2y+3)]$
$=\mathbf{(3x+y+2)(x+2y+3)}$

(3) $2x^2-2y^2-z^2+3yz+zx-3xy$
$=2x^2-(3y-z)x-(2y^2-3yz+z^2)$
$=2x^2-(3y-z)x-(2y-z)(y-z)$
$=[2x+(y-z)][x-(2y-z)]$
$=\mathbf{(2x+y-z)(x-2y+z)}$

(4) $2a^2+6b^2-18c^2+3bc-7ab$
$=2a^2-7ba+3(2b^2+bc-6c^2)$
$=2a^2-7ba+3(2b-3c)(b+2c)$
$=[2a-3(b+2c)][a-(2b-3c)]$
$=\mathbf{(2a-3b-6c)(a-2b+3c)}$

J50b

（5） $a^2+5ab+5a-6b^2-12b-6$

$=a^2+5(b+1)a-6(b^2+2b+1)$

$=a^2+5(b+1)a-6(b+1)^2$

$=[a+6(b+1)][a-(b+1)]$

$=\boldsymbol{(a+6b+6)(a-b-1)}$

（6） $a^2-4b^2-3c^2+8bc-2ac$

$=a^2-2ca-(4b^2-8bc+3c^2)$

$=a^2-2ca-(2b-3c)(2b-c)$

$=[a+(2b-3c)][a-(2b-c)]$

$=\boldsymbol{(a+2b-3c)(a-2b+c)}$

（7） $a^2-4b^2+4bc-c^2$

$=a^2-(4b^2-4bc+c^2)$

$=a^2-(2b-c)^2$

$=[a+(2b-c)][a-(2b-c)]$

$=\boldsymbol{(a+2b-c)(a-2b+c)}$

（8） $1-x^2+x^3-x^5$

$=(1-x^2)+x^3(1-x^2)$

$=(1-x^2)(1+x^3)$

$=(1+x)(1-x)(1+x)(1-x+x^2)$

$=\boldsymbol{(1+x)^2(1-x)(1-x+x^2)}$

J51a KUMON

Factorisation V

Factorise the following expressions.

Ex.

$$x^4 - 6x^2 + 1$$

Break up $-6x^2$ into two terms, $-2x^2$ and $-4x^2$.

$$= x^4 - 2x^2 + 1 - 4x^2$$

Factorise $x^4 - 2x^2 + 1$ and rewrite $-4x^2$ as $-(2x)^2$.

$$= (x^2 - 1)^2 - (2x)^2$$

Factorise $(\quad)^2 - (\quad)^2$.

$$= (x^2 - 1 + 2x)(x^2 - 1 - 2x)$$

$$= (x^2 + 2x - 1)(x^2 - 2x - 1)$$

(1) $x^4 - 11x^2 + 1 = (x^4 - 2x^2 + 1) - 9x^2$

$= (x^2 - 1)^2 - (3x)^2$

$= (x^2 - 1 + 3x)(x^2 - 1 - 3x)$

$= \boldsymbol{(x^2 + 3x - 1)(x^2 - 3x - 1)}$

(2) $x^4 - 27x^2 + 1 = (x^4 - 2x^2 + 1) - 25x^2$

$= (x^2 - 1)^2 - (5x)^2$

$= (x^2 - 1 + 5x)(x^2 - 1 - 5x)$

$= \boldsymbol{(x^2 + 5x - 1)(x^2 - 5x - 1)}$

(3) $x^4 - 7x^2 + 1 = (x^4 + 2x^2 + 1) - \boxed{\boldsymbol{9x^2}}$

$= (x^2 + 1)^2 - (3x)^2$

$= (x^2 + 1 + 3x)(x^2 + 1 - 3x)$

$= \boldsymbol{(x^2 + 3x + 1)(x^2 - 3x + 1)}$

J5Ib

（４） $x^4-23x^2+1=(x^4+2x^2+1)-25x^2$
$$=(x^2+1)^2-(5x)^2$$
$$=(x^2+1+5x)(x^2+1-5x)$$
$$=\boldsymbol{(x^2+5x+1)(x^2-5x+1)}$$

（５） $a^4-13a^2+4=(a^4-4a^2+4)-9a^2$
$$=(a^2-2)^2-(3a)^2$$
$$=(a^2-2+3a)(a^2-2-3a)$$
$$=\boldsymbol{(a^2+3a-2)(a^2-3a-2)}$$

（６） $x^4+2x^2+9=(x^4+6x^2+9)-4x^2$
$$=(x^2+3)^2-(2x)^2$$
$$=(x^2+3+2x)(x^2+3-2x)$$
$$=\boldsymbol{(x^2+2x+3)(x^2-2x+3)}$$

（７） $x^4+x^2+1=(x^4+2x^2+1)-x^2$
$$=(x^2+1)^2-x^2$$
$$=(x^2+1+x)(x^2+1-x)$$
$$=\boldsymbol{(x^2+x+1)(x^2-x+1)}$$

（８） $a^4+a^2b^2+b^4=(a^4+2a^2b^2+b^4)-a^2b^2$
$$=(a^2+b^2)^2-(ab)^2$$
$$=(a^2+b^2+ab)(a^2+b^2-ab)$$
$$=\boldsymbol{(a^2+ab+b^2)(a^2-ab+b^2)}$$

J 52a

KUMON

Factorisation V

Factorise the following expressions.

(1) $$\begin{aligned} x^4-18x^2+1 &= (x^4-2x^2+1)-16x^2 \\ &= (x^2-1)^2-(4x)^2 \\ &= [(x^2-1)+4x][(x^2-1)-4x] \\ &= \boldsymbol{(x^2+4x-1)(x^2-4x-1)} \end{aligned}$$

(2) $$\begin{aligned} x^4-14x^2+1 &= (x^4+2x^2+1)-16x^2 \\ &= (x^2+1)^2-(4x)^2 \\ &= [(x^2+1)+4x][(x^2+1)-4x] \\ &= \boldsymbol{(x^2+4x+1)(x^2-4x+1)} \end{aligned}$$

(3) $$\begin{aligned} x^4-38x^2+1 &= (x^4-2x^2+1)-36x^2 \\ &= (x^2-1)^2-(6x)^2 \\ &= [(x^2-1)+6x][(x^2-1)-6x] \\ &= \boldsymbol{(x^2+6x-1)(x^2-6x-1)} \end{aligned}$$

(4) $$\begin{aligned} x^4-34x^2+1 &= (x^4+2x^2+1)-36x^2 \\ &= (x^2+1)^2-(6x)^2 \\ &= [(x^2+1)+6x][(x^2+1)-6x] \\ &= \boldsymbol{(x^2+6x+1)(x^2-6x+1)} \end{aligned}$$

J52b

(5) $4x^4+3x^2+1=(4x^4+4x^2+1)-x^2$

$$=(2x^2+1)^2-x^2$$
$$=[(2x^2+1)+x][(2x^2+1)-x]$$
$$=\mathbf{(2x^2+x+1)(2x^2-x+1)}$$

(6) $4x^4+11x^2+9=(4x^4+12x^2+9)-x^2$

$$=(2x^2+3)^2-x^2$$
$$=[(2x^2+3)+x][(2x^2+3)-x]$$
$$=\mathbf{(2x^2+x+3)(2x^2-x+3)}$$

(7)* x^4-10x^2+9

The answer has four factors.

$$=(x^4-6x^2+9)-4x^2$$
$$=(x^2-3)^2-(2x)^2$$
$$=[(x^2-3)+2x][(x^2-3)-2x]$$
$$=(x^2+2x-3)(x^2-2x-3)$$
$$=\mathbf{(x+3)(x-1)(x-3)(x+1)}$$

Alternative Solution

$$x^4-10x^2+9$$
$$=(x^2-9)(x^2-1)$$
$$=\mathbf{(x+3)(x-3)(x+1)(x-1)}$$

(8) $9x^4-13x^2+4=(9x^4-12x^2+4)-x^2$

$$=(3x^2-2)^2-x^2$$
$$=[(3x^2-2)+x][(3x^2-2)-x]$$
$$=(3x^2+x-2)(3x^2-x-2)$$
$$=\mathbf{(3x-2)(x+1)(3x+2)(x-1)}$$

J 53a KUMON

Factorisation V

Factorise the following expressions.

Ex.

$(\underset{\sim\sim}{x^2+x}-6)(\underset{\sim\sim}{x^2+x}-2)+3$ 〰 is repeated.

$=(A-6)(A-2)+3$ [where $A=x^2+x$]

$=A^2-8A+15$

$=(A-5)(A-3)$

Substitute A with x^2+x.

$=(x^2+x-5)(x^2+x-3)$

(1) $(\underset{\sim\sim}{x^2+x}-7)(\underset{\sim\sim}{x^2+x}-5)-8$

$=(A-7)(A-5)-8$ [where $A=x^2+x$]

$=A^2-12A+27$

$=(A-9)(A-3)$

$=\boldsymbol{(x^2+x-9)(x^2+x-3)}$

Remember to express your answer in terms of x (or whatever variable is used in the original question).

(2) $(\underset{\sim\sim}{x^2+2x}-8)(\underset{\sim\sim}{x^2+2x}+1)-10$

$=(A-8)(A+1)-10$ [where $A=x^2+2x$]

$=A^2-7A-18$

$=(A-9)(A+2)$

$=\boldsymbol{(x^2+2x-9)(x^2+2x+2)}$

(3) $(\underset{\sim\sim}{x^2+5x})(\underset{\sim\sim}{x^2+5x}+6)-16$

$=A(A+6)-16$ [where $A=x^2+5x$]

$=A^2+6A-16$

$=(A+8)(A-2)$

$=\boldsymbol{(x^2+5x+8)(x^2+5x-2)}$

J53b

Ex.

$$(x^2+x-5)(x^2+2x-5)-12x^2$$ is repeated.

$=(A+x)(A+2x)-12x^2$ [where $A=x^2-5$]

$=A^2+3xA-10x^2$

$=(A+5x)(A-2x)$

$=(x^2+5x-5)(x^2-2x-5)$

(4) $(x^2-2x-7)(x^2+3x-7)-6x^2$

$=(A-2x)(A+3x)-6x^2$ [where $A=x^2-7$]

$=A^2+xA-12x^2$

$=(A+4x)(A-3x)$

$=\mathbf{(x^2+4x-7)(x^2-3x-7)}$

(5) $(x^2+3x+3)(x^2-4x+3)+6x^2$

$=(A+3x)(A-4x)+6x^2$ [where $A=x^2+3$]

$=A^2-xA-6x^2$

$=(A-3x)(A+2x)$

$=\mathbf{(x^2-3x+3)(x^2+2x+3)}$

(6)* $(x^2-7x-18)(x^2+3x-18)+24x^2$

$=(A-7x)(A+3x)+24x^2$ [where $A=x^2-18$]

$=A^2-4xA+3x^2$

$=(A-3x)(A-x)$

$=(x^2-3x-18)(x^2-x-18)$

$=\mathbf{(x-6)(x+3)(x^2-x-18)}$

Make sure your answer is factorised completely.

J 54 a KUMON

Factorisation V

Factorise the following expressions.

Ex.

$$(x+1)(x+2)(x+3)(x+4)-8$$
$$=(x^2+5x+6)(x^2+5x+4)-8$$
$$=(x^2+5x)^2+10(x^2+5x)+16$$
$$=(x^2+5x+8)(x^2+5x+2)$$

Multiply out $(x+2)(x+3)$ and $(x+1)(x+4)$.

Treat x^2+5x as a single unit (or substitute $x^2+5x=A$).

(1) $(x-1)(x-2)(x-3)(x-4)-15$
$=(x-1)(x-4)(x-2)(x-3)-15$
$=(x^2-5x+4)(x^2-5x+6)-15$
$=(x^2-5x)^2+10(x^2-5x)+9$
$=\boldsymbol{(x^2-5x+9)(x^2-5x+1)}$

Note:
In these questions, consider carefully which terms to multiply together. The aim is to form two brackets with the same unit at the start of each, e.g.

$(\underline{x^2-5x}+4)(\underline{x^2-5x}+6)-15$

or

$(\underline{x^2+5x}+6)(\underline{x^2+5x}+4)-3$

(2) $(x+1)(x+2)(x+3)(x+4)-3$
$=(x+2)(x+3)(x+1)(x+4)-3$
$=(x^2+5x+6)(x^2+5x+4)-3$
$=(x^2+5x)^2+10(x^2+5x)+21$
$=\boldsymbol{(x^2+5x+7)(x^2+5x+3)}$

J54b

(3) $(x-1)(x-3)(x+2)(x+4)+24$

$= (x-1)(x+2)(x-3)(x+4)+24$

$= (x^2+x-2)(x^2+x-12)+24$

$= (x^2+x)^2-14(x^2+x)+48$

$= (x^2+x-8)(x^2+x-6)$

$= \boldsymbol{(x^2+x-8)(x+3)(x-2)}$

Make sure your answer
is factorised completely.

(4) $x(x+1)(x+2)(x+3)-15$

$= x(x+3)(x+1)(x+2)-15$

$= (x^2+3x)(x^2+3x+2)-15$

$= (x^2+3x)^2+2(x^2+3x)-15$

$= \boldsymbol{(x^2+3x+5)(x^2+3x-3)}$

(5) $(x+1)(x+2)(x+3)(x+4)+1$

$= (x+1)(x+4)(x+2)(x+3)+1$

$= (x^2+5x+4)(x^2+5x+6)+1$

$= (x^2+5x)^2+10(x^2+5x)+25$

$= \boldsymbol{(x^2+5x+5)^2}$

J 55a KUMON

Factorisation V

Factorise the following expressions.

Ex.

$$x^2(b-c)+\underline{b^2(c-x)}+\underline{c^2(x-b)}$$

Expand the underlined terms.

$$=(b-c)x^2+b^2c-b^2x+c^2x-c^2b$$

Arrange in powers of x.

$$=(b-c)x^2-(b^2-c^2)x+bc(b-c)$$

$$=(b-c)x^2-(b+c)(b-c)x+bc(b-c)$$

Take out the common factor $(b-c)$.

$$=(b-c)[x^2-(b+c)x+bc]$$

$$=(b-c)(x-b)(x-c)$$

(1) $a^2(b-c)+b^2(c-a)+c^2(a-b)$

$=(b-c)a^2+b^2c-b^2a+c^2a-c^2b$

$=(b-c)a^2-(b^2-c^2)a+bc(b-c)$

$=(b-c)a^2-(b+c)(b-c)a+bc(b-c)$

$=(b-c)[a^2-(b+c)a+bc]$

$=\boldsymbol{(b-c)(a-b)(a-c)}$

The key concept in these exercises is to expand the polynomial and arrange it in descending powers of a particular variable, in this case a, before factorising.

(2) $x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz$

$=(y+z)x^2+y^2z+y^2x+z^2x+z^2y+2xyz$

$=(y+z)x^2+(y^2+2yz+z^2)x+yz(y+z)$

$=(y+z)x^2+(y+z)^2x+yz(y+z)$

$=(y+z)[x^2+(y+z)x+yz]$

$=\boldsymbol{(y+z)(x+y)(x+z)}$

J55b

（3） $bc(b-c)-ca(c+a)+ab(a+b)$

$=bc(b-c)-c^2a-ca^2+a^2b+ab^2$

$=(b-c)a^2+(b^2-c^2)a+bc(b-c)$

$=(b-c)a^2+(b+c)(b-c)a+bc(b-c)$

$=(b-c)[a^2+(b+c)a+bc]$

$=\boldsymbol{(b-c)(a+b)(a+c)}$

（4） $a^2(b+c)+b^2(c-a)+c^2(b-a)-2abc$

$=(b+c)a^2+b^2c-b^2a+c^2b-c^2a-2abc$

$=(b+c)a^2-(b^2+2bc+c^2)a+bc(b+c)$

$=(b+c)a^2-(b+c)^2a+bc(b+c)$

$=(b+c)[a^2-(b+c)a+bc]$

$=\boldsymbol{(b+c)(a-b)(a-c)}$

J 56a KUMON

Factorisation V

Factorise the following expressions.

(1) $x^2(b-c)+b^2(c+x)-c^2(x+b)$
$=(b-c)x^2+b^2c+b^2x-c^2x-c^2b$
$=(b-c)x^2+(b^2-c^2)x+bc(b-c)$
$=(b-c)x^2+(b+c)(b-c)x+bc(b-c)$
$=(b-c)[x^2+(b+c)x+bc]$
$=\boldsymbol{(b-c)(x+b)(x+c)}$

(2) $x(b^2-c^2)+b(c^2-x^2)+c(x^2-b^2)$
$=x(b^2-c^2)+bc^2-bx^2+cx^2-cb^2$
$=-(b-c)x^2+(b^2-c^2)x-bc(b-c)$
$=-(b-c)x^2+(b+c)(b-c)x-bc(b-c)$
$=-(b-c)[x^2-(b+c)x+bc]$
$=\boldsymbol{-(b-c)(x-b)(x-c)}$

Alternative Solution
$x(b^2-c^2)+b(c^2-x^2)+c(x^2-b^2)$
$=x(b^2-c^2)+bc^2-bx^2+cx^2-cb^2$
$=(c-b)x^2-(c^2-b^2)x+bc(c-b)$
$=(c-b)x^2-(c+b)(c-b)x+bc(c-b)$
$=(c-b)[x^2-(b+c)x+bc]$
$=\boldsymbol{(c-b)(x-b)(x-c)}$

(3) $x^2(y+z)+y^2(z-x)+z^2(y-x)-2xyz$
$=(y+z)x^2+y^2z-y^2x+z^2y-z^2x-2xyz$
$=(y+z)x^2-(y^2+2yz+z^2)x+yz(y+z)$
$=(y+z)[x^2-(y+z)x+yz]$
$=\boldsymbol{(y+z)(x-y)(x-z)}$

J56b

(4) $(x+y+z)(yz+zx+xy)-xyz$

$$
\begin{aligned}
&=[x+(y+z)][(y+z)x+yz]-xyz \\
&=(y+z)x^2+xyz+(y+z)^2x+yz(y+z)-xyz \\
&=(y+z)x^2+(y+z)^2x+yz(y+z) \\
&=(y+z)[x^2+(y+z)x+yz] \\
&=\boldsymbol{(y+z)(x+y)(x+z)}
\end{aligned}
$$

In this step, treating $(y+z)$ as a single unit makes the expansion much simpler.

(5) $a^2b-a^2c-ac^2-ab^2-b^2c+bc^2+2abc$

$$
\begin{aligned}
&=(b-c)a^2-(b^2-2bc+c^2)a-bc(b-c) \\
&=(b-c)a^2-(b-c)^2a-bc(b-c) \\
&=(b-c)[a^2-(b-c)a-bc] \\
&=\boldsymbol{(b-c)(a-b)(a+c)}
\end{aligned}
$$

J 57a KUMON

Factorisation V

Factorise the following expressions.

Ex.

$x^2-y^2+2yz-2zx-4x+2y+2z+3$

Arrange in powers of x.

[Sol]

$$\begin{aligned}&x^2-y^2+2yz-2zx-4x+2y+2z+3\\&=x^2+(-2z-4)x-[y^2-(2z+2)y-(2z+3)]\\&=x^2+(-2z-4)x-(y+1)[y-(2z+3)]\\&=[x-(y+1)][x+(y-2z-3)]\\&=(x-y-1)(x+y-2z-3)\end{aligned}$$

Arrange in descending powers of x.

(1) $x^2-y^2+2zx+2yz+2y-2z-1$

Arrange in powers of x.

[Sol]

$$\begin{aligned}&x^2-y^2+2zx+2yz+2y-2z-1\\&=x^2+2zx-[y^2-(2z+2)y+(2z+1)]\\&=x^2+2zx-(y-1)[y-(2z+1)]\\&=[x+(y-1)][x-(y-2z-1)]\\&=\boldsymbol{(x+y-1)(x-y+2z+1)}\end{aligned}$$

See foot note.

Note: To factorise $y^2-(2z+2)y+(2z+1)$, look for two terms that multiply to $(2z+1)$ and add to $-(2z+2)$. The terms are -1 and $-(2z+1)$.

J57b

Ex.

$x^2-y^2+2yz-2zx-4x+2y+2z+3$

Arrange in powers of z.

[Sol]

$$
\begin{aligned}
&x^2-y^2+2yz-2zx-4x+2y+2z+3 \\
&=2(y-x+1)z+(x^2-y^2-4x+2y+3) \\
&=2(y-x+1)z+[x^2-4x-(y^2-2y-3)] \\
&=2(y-x+1)z+[x^2-4x-(y-3)(y+1)] \\
&=2(y-x+1)z+[x+(y-3)][x-(y+1)] \\
&=-2(x-y-1)z+(x+y-3)(x-y-1) \\
&=(x-y-1)(x+y-2z-3)
\end{aligned}
$$

Arrange in powers of z.

Take out the common factor $(x-y-1)$.

(2) $x^2-y^2+2zx+2yz+2y-2z-1$

Arrange in powers of z.

[Sol]

$$
\begin{aligned}
&x^2-y^2+2zx+2yz+2y-2z-1 \\
&=2(x+y-1)z+(x^2-y^2+2y-1) \\
&=2(x+y-1)z+[x^2-(y^2-2y+1)] \\
&=2(x+y-1)z+[x^2-(y-1)^2] \\
&=2(x+y-1)z+[x+(y-1)][x-(y-1)] \\
&=2(x+y-1)z+(x+y-1)(x-y+1) \\
&=(x+y-1)[2z+(x-y+1)] \\
&=\boldsymbol{(x+y-1)(x-y+2z+1)}
\end{aligned}
$$

J 58a KUMON

Factorisation V

1. $ab+2ac+3b^2+6bc-5a-13b+4c-10$

Arrange (1) in powers of a,

(2) in powers of b,

and factorise.

(1) In powers of a,

You will get a linear expression.

$$
\begin{aligned}
& ab+2ac+3b^2+6bc-5a-13b+4c-10 \\
&= (b+2c-5)a+(3b^2+6bc-13b+4c-10) \\
&= (b+2c-5)a+[3b^2+(6c-13)b+2(2c-5)] \\
&= (b+2c-5)a+(3b+2)(b+2c-5) \\
&= (b+2c-5)[a+(3b+2)] \\
&= \boldsymbol{(b+2c-5)(a+3b+2)}
\end{aligned}
$$

(2) In powers of b,

You will get a quadratic expression.

$$
\begin{aligned}
& ab+2ac+3b^2+6bc-5a-13b+4c-10 \\
&= 3b^2+(a+6c-13)b+(2ac-5a+4c-10) \\
&= 3b^2+(a+6c-13)b+a(2c-5)+2(2c-5) \\
&= 3b^2+(a+6c-13)b+(2c-5)(a+2) \\
&= [3b+(a+2)][b+(2c-5)] \\
&= (3b+a+2)(b+2c-5) \\
&= \boldsymbol{(a+3b+2)(b+2c-5)}
\end{aligned}
$$

J58b

2. $a^2+3b^2+4ab+2ac+6bc-4b+4c-4$
   Arrange in powers of a, b or c, then factorise.

In powers of a,

$$\begin{aligned}
&a^2+3b^2+4ab+2ac+6bc-4b+4c-4\\
&=a^2+2(2b+c)a+[3b^2+2(3c-2)b+2\cdot2(c-1)]\\
&=a^2+2(2b+c)a+(3b+2)[b+2(c-1)]\\
&=a^2+2(2b+c)a+(3b+2)(b+2c-2)\\
&=[a+(3b+2)][a+(b+2c-2)]\\
&=\boldsymbol{(a+3b+2)(a+b+2c-2)}
\end{aligned}$$

In powers of b,

$$\begin{aligned}
&a^2+3b^2+4ab+2ac+6bc-4b+4c-4\\
&=3b^2+2(2a+3c-2)b+(a^2+2ac+4c-4)\\
&=3b^2+2(2a+3c-2)b+(a+2)(a-2)+2c(a+2)\\
&=3b^2+2(2a+3c-2)b+(a+2)(a-2+2c)\\
&=[3b+(a+2)][b+(a-2+2c)]\\
&=(3b+a+2)(b+a-2+2c)\\
&=\boldsymbol{(a+3b+2)(a+b+2c-2)}
\end{aligned}$$

In powers of c,

$$\begin{aligned}
&a^2+3b^2+4ab+2ac+6bc-4b+4c-4\\
&=2(a+3b+2)c+(a^2+3b^2+4ab-4b-4)\\
&=2(a+3b+2)c+[a^2+4ba+(3b+2)(b-2)]\\
&=2(a+3b+2)c+(a+3b+2)(a+b-2)\\
&=(a+3b+2)(2c+a+b-2)\\
&=\boldsymbol{(a+3b+2)(a+b+2c-2)}
\end{aligned}$$

J 59a KUMON

Factorisation V

Factorise the following expressions.

(1) $x^3+(2a+1)x^2+(a^2+2a-1)x+(a^2-1)$

Arrange in powers of a.

$=x^3+2ax^2+x^2+a^2x+2ax-x+a^2-1$

$=(x+1)a^2+2x(x+1)a+x^3+x^2-x-1$

$=(x+1)a^2+2x(x+1)a+x^2(x+1)-(x+1)$

$=(x+1)[a^2+2xa+(x^2-1)]$

$=(x+1)[a^2+2xa+(x+1)(x-1)]$

$=(x+1)[a+(x+1)][a+(x-1)]$

$\boldsymbol{=(x+1)(a+x+1)(a+x-1)}$

$[\boldsymbol{=(x+1)(x+a+1)(x+a-1)}]$

J59b

(2) $ax^2-a^3-a^2b+ab^2+b^3-bx^2$

Arrange in powers of x.

$=(a-b)x^2-(a^3-b^3)-ab(a-b)$

Use the formula: $a^3-b^3=(a-b)(a^2+ab+b^2)$

$=(a-b)x^2-(a-b)(a^2+ab+b^2)-ab(a-b)$

$=(a-b)[x^2-(a^2+ab+b^2)-ab]$

$=(a-b)[x^2-(a^2+2ab+b^2)]$

$=(a-b)[x^2-(a+b)^2]$

$=\boldsymbol{(a-b)(x+a+b)(x-a-b)}$

J 60a

KUMON

Factorisation V

Factorise the following expressions.

(1) $x^2 - y^2 + yz - zx - 4x + 2y + z + 3$

$= x^2 - (z+4)x - [y^2 - (z+2)y - (z+3)]$

$= x^2 - (z+4)x - (y+1)(y-z-3)$

$= [x-(y+1)][x+(y-z-3)]$

$= \boldsymbol{(x-y-1)(x+y-z-3)}$

(2) $ac^2 - a^3 - a^2b + ab^2 + b^3 - bc^2$

$= (a-b)c^2 - ab(a-b) - (a^3 - b^3)$

$= (a-b)c^2 - ab(a-b) - (a-b)(a^2+ab+b^2)$

$= (a-b)[c^2 - ab - (a^2+ab+b^2)]$

$= (a-b)[c^2 - (a^2+2ab+b^2)]$

$= (a-b)[c^2 - (a+b)^2]$

$= \boldsymbol{(a-b)(c+a+b)(c-a-b)}$

$\left(= \boldsymbol{-(a-b)(a+b+c)(a+b-c)}\right)$

J60b

(3) $(a+b-c)(ab-bc-ca)+abc$

$=[a+(b-c)][(b-c)a-bc]+abc$

$=(b-c)a^2+[(b-c)^2-bc]a-(b-c)bc+abc$

$=(b-c)a^2+[(b-c)^2-bc+bc]a-(b-c)bc$

$=(b-c)a^2+(b-c)^2a-(b-c)bc$

$=(b-c)[a^2+(b-c)a-bc]$

$=\boldsymbol{(b-c)(a+b)(a-c)}$

(4) $(xy-1)(x-1)(y+1)-xy$

$=(y+1)[yx^2-(y+1)x+1]-xy$

$=y(y+1)x^2-[(y+1)^2+y]x+(y+1)$

$=[yx-(y+1)][(y+1)x-1]$

$=\boldsymbol{(xy-y-1)(xy+x-1)}$

Alternative Solution

$(xy-1)(x-1)(y+1)-xy$

$=(xy-1)[(xy-1)+(x-y)]-xy$

$=(xy-1)^2+(x-y)(xy-1)-xy$

$=[(xy-1)+x][(xy-1)-y]$

$=(xy-1+x)(xy-1-y)$

$=\boldsymbol{(xy+x-1)(xy-y-1)}$

J61a KUMON

Fractional Expressions

Reduce the following fractional expressions.

Ex.

$$\frac{2a^2}{3a^3} = \frac{2}{3a} \qquad \frac{2a^3b}{-4a^2b^5} = -\frac{a}{2b^4}$$

(1) $\dfrac{6ab^5}{9a^3b^2} = \boldsymbol{\dfrac{2b^3}{3a^2}}$

(2) $\dfrac{28a^3x^4}{35a^3x^5} = \boldsymbol{\dfrac{4}{5x}}$

(3) $\dfrac{ax}{a^2x^3} = \boldsymbol{\dfrac{1}{ax^2}}$

(4) $\dfrac{18a^3}{24a^5} = \boldsymbol{\dfrac{3}{4a^2}}$

(5) $\dfrac{2x^2y}{-6yz^2} = \boldsymbol{-\dfrac{x^2}{3z^2}}$

(6) $\dfrac{-6xz^2}{3xyz} = \boldsymbol{-\dfrac{2z}{y}}$

J61b

Ex.

$$\frac{x^2-25}{x^2-3x-10}=\frac{(x+5)(x-5)}{(x-5)(x+2)}=\frac{x+5}{x+2}$$ ☜ Cancel $(x-5)$.

$$\frac{x+1}{x^2-x-2}=\frac{x+1}{(x-2)(x+1)}=\frac{1}{x-2}$$ ☜ Cancel $(x+1)$.

(7) $$\frac{x^2-5x+6}{x^2-7x+12}=\frac{(x-3)(x-2)}{(x-4)(x-3)}=\boldsymbol{\frac{x-2}{x-4}}$$

(8) $$\frac{x^2-x-20}{2x^2-7x-15}=\frac{(x-5)(x+4)}{(2x+3)(x-5)}=\boldsymbol{\frac{x+4}{2x+3}}$$

(9) $$\frac{x+2}{x^2+x-2}=\frac{x+2}{(x+2)(x-1)}=\boldsymbol{\frac{1}{x-1}}$$

(10) $$\frac{x^2-6x+9}{x^2-x-6}=\frac{(x-3)^2}{(x-3)(x+2)}=\boldsymbol{\frac{x-3}{x+2}}$$

(11) $$\frac{3a^2+6a}{a^2+4a+4}=\frac{3a(a+2)}{(a+2)^2}=\boldsymbol{\frac{3a}{a+2}}$$

J 62a KUMON

Fractional Expressions

Reduce the following fractional expressions.

Ex.

$$\frac{b-a}{(a-b)(x-3)}=\frac{-(a-b)}{(a-b)(x-3)}=-\frac{1}{x-3}$$

$$\frac{-x^2+5x-6}{x^2-7x+12}=\frac{-(x-3)(x-2)}{(x-4)(x-3)}=-\frac{x-2}{x-4}$$

(1) $\dfrac{x^2+5x-14}{4-x^2}=\dfrac{(x+7)(x-2)}{-(x+2)(x-2)}=\boldsymbol{-\dfrac{x+7}{x+2}}$

(2) $\dfrac{a^2b(x-a)}{ab^2(a-x)}=\dfrac{a^2b(x-a)}{-ab^2(x-a)}=\boldsymbol{-\dfrac{a}{b}}$

(3) $\dfrac{(x-5)^2}{(5+x)(5-x)}=\dfrac{(x-5)^2}{-(x+5)(x-5)}=\boldsymbol{-\dfrac{x-5}{x+5}}$

(4) $\dfrac{m-mx}{x^2-2x+1}=\dfrac{m(1-x)}{(x-1)^2}=\dfrac{-m(x-1)}{(x-1)^2}=\boldsymbol{-\dfrac{m}{x-1}}$

(5) $\dfrac{x^2-1}{(1+x)^2}=\dfrac{(x+1)(x-1)}{(x+1)^2}=\boldsymbol{\dfrac{x-1}{x+1}}$

J62b

(6) $\dfrac{x^2-(y+z)^2}{z^2-(x+y)^2}$

$$=\frac{(x+y+z)(x-y-z)}{(z+x+y)(z-x-y)}=\frac{(x+y+z)(x-y-z)}{-(x+y+z)(x+y-z)}=\boldsymbol{-\frac{x-y-z}{x+y-z}}$$

(7) $\dfrac{(b-c)^2-a^2}{(c+a)^2-b^2}$

$$=\frac{(b-c+a)(b-c-a)}{(c+a+b)(c+a-b)}=\frac{-(a+b-c)(a-b+c)}{(a+b+c)(a-b+c)}=\boldsymbol{-\frac{a+b-c}{a+b+c}}$$

(8) $\dfrac{x^3+3x^2y+3xy^2+y^3}{x^3+y^3}$

$$=\frac{(x+y)^3}{(x+y)(x^2-xy+y^2)}=\frac{(x+y)(x+y)^2}{(x+y)(x^2-xy+y^2)}=\boldsymbol{\frac{(x+y)^2}{x^2-xy+y^2}}$$

(9) $\dfrac{x^3-3x^2y+3xy^2-y^3}{x^3-y^3}$

$$=\frac{(x-y)^3}{(x-y)(x^2+xy+y^2)}=\frac{(x-y)(x-y)^2}{(x-y)(x^2+xy+y^2)}=\boldsymbol{\frac{(x-y)^2}{x^2+xy+y^2}}$$

(10) $\dfrac{a^2-ab+b^2}{a^3+b^3}$

$$=\frac{a^2-ab+b^2}{(a+b)(a^2-ab+b^2)}=\boldsymbol{\frac{1}{a+b}}$$

J 63a KUMON

Fractional Expressions

Reduce the following fractional expressions.

(1) $\dfrac{ax+ay+bx+by}{ax-by-bx+ay}$

$= \dfrac{a(x+y)+b(x+y)}{a(x+y)-b(x+y)} = \dfrac{(x+y)(a+b)}{(x+y)(a-b)} = \boldsymbol{\dfrac{a+b}{a-b}}$

(2) $\dfrac{x^2-xy-cy+cx}{x^2+bx-xy-by}$

$= \dfrac{x(x-y)+c(x-y)}{x(x-y)+b(x-y)} = \dfrac{(x-y)(x+c)}{(x-y)(x+b)} = \boldsymbol{\dfrac{x+c}{x+b}}$

(3) $\dfrac{x^2+7x+12}{x^3+4x^2-9x-36}$

$= \dfrac{(x+4)(x+3)}{x^2(x+4)-9(x+4)} = \dfrac{(x+4)(x+3)}{(x+4)(x^2-9)}$

$= \dfrac{(x+4)(x+3)}{(x+4)(x+3)(x-3)} = \boldsymbol{\dfrac{1}{x-3}}$

(4) $\dfrac{a^2-ab+ab^2}{a^2-b^2+ab^2+b^3}$

$= \dfrac{a(a-b+b^2)}{(a+b)(a-b)+b^2(a+b)} = \dfrac{a(a-b+b^2)}{(a+b)(a-b+b^2)} = \boldsymbol{\dfrac{a}{a+b}}$

J63b

（5）$\dfrac{a^2-ab+ac-bc}{a^3-b^3}$

$$=\frac{a(a-b)+c(a-b)}{(a-b)(a^2+ab+b^2)}=\frac{(a-b)(a+c)}{(a-b)(a^2+ab+b^2)}=\boldsymbol{\frac{a+c}{a^2+ab+b^2}}$$

（6）$\dfrac{x^3-y^3}{x^4+x^2y^2+y^4}$

$$=\frac{(x-y)(x^2+xy+y^2)}{(x^2+y^2)^2-x^2y^2}=\frac{(x-y)(x^2+xy+y^2)}{(x^2+xy+y^2)(x^2-xy+y^2)}$$

$$=\boldsymbol{\frac{x-y}{x^2-xy+y^2}}$$

（7）$\dfrac{x^3-2x^2y+xy^2}{x^2-3xy+2y^2}$

$$=\frac{x(x^2-2xy+y^2)}{(x-2y)(x-y)}=\frac{x(x-y)^2}{(x-2y)(x-y)}=\boldsymbol{\frac{x(x-y)}{x-2y}}$$

J 64 a KUMON

Fractional Expressions

Simplify the following fractional expressions.

Ex.

$$\frac{3ax^2}{b^4} \times \frac{2a^3b^2}{9xy} = \frac{2a^4x}{3b^2y}$$

(1) $\dfrac{2a^2}{3bc} \times \dfrac{5b^3c^3}{8a^3x} = \boldsymbol{\dfrac{5b^2c^2}{12ax}}$

(2) $\dfrac{x^2}{yz} \times \dfrac{y^2}{zx} \times \dfrac{z^2}{xy} = \mathbf{1}$

(3) $\dfrac{4a^2b}{5x^2y} \div \dfrac{2ab^2}{15xy^2} = \dfrac{4a^2b}{5x^2y} \times \dfrac{15xy^2}{2ab^2} = \boldsymbol{\dfrac{6ay}{bx}}$

(4) $\dfrac{2a^2}{x^2y^2} \div \dfrac{-a}{xy} \times \dfrac{x^3}{2a^3} = -\dfrac{2a^2}{x^2y^2} \times \dfrac{xy}{a} \times \dfrac{x^3}{2a^3} = \boldsymbol{-\dfrac{x^2}{a^2y}}$

(5) $\dfrac{b^3}{a} \times \dfrac{b^2}{2a^4} \div \left(-\dfrac{4a^6}{b^6}\right) = -\dfrac{b^3}{a} \times \dfrac{b^2}{2a^4} \times \dfrac{b^6}{4a^6} = \boldsymbol{-\dfrac{b^{11}}{8a^{11}}}$

J64b

(6) $\dfrac{xy}{z^2} \div \dfrac{x^4}{y^2z^2} \div \dfrac{y^6}{z^3x^3}$

$= \dfrac{xy}{z^2} \times \dfrac{y^2z^2}{x^4} \times \dfrac{z^3x^3}{y^6} = \boldsymbol{\dfrac{z^3}{y^3}}$

(7) $\dfrac{(-4a^2b)^2}{21x^3y^2} \times \dfrac{3x^2y}{(2a)^2b}$

$= \dfrac{16a^4b^2}{21x^3y^2} \times \dfrac{3x^2y}{4a^2b} = \boldsymbol{\dfrac{4a^2b}{7xy}}$

(8) $\dfrac{(7a^2b)^2}{21x^3y^2} \div \dfrac{3x^2y}{35(-ab)^3}$

$= \dfrac{49a^4b^2}{21x^3y^2} \times \left(-\dfrac{35a^3b^3}{3x^2y}\right) = \boldsymbol{-\dfrac{245a^7b^5}{9x^5y^3}}$

(9) $\dfrac{(-3x^4y)^2}{4a^2b^3} \times \dfrac{8a^3b^2}{(-3xy^2)^3} \div \dfrac{(2a^2x^3)^2}{9b^3y^5}$

$= \dfrac{9x^8y^2}{4a^2b^3} \times \left(-\dfrac{8a^3b^2}{27x^3y^6}\right) \times \dfrac{9b^3y^5}{4a^4x^6} = \boldsymbol{-\dfrac{3b^2y}{2a^3x}}$

J 65a KUMON

Fractional Expressions

Simplify the following fractional expressions.

(1) $\dfrac{x^2-1}{3}\times\dfrac{6a}{x-1}$

$$=\frac{(x+1)(x-1)}{3}\times\frac{6a}{x-1}=\boldsymbol{2a(x+1)}$$

(2) $\dfrac{x^2+2x}{x^2-9}\times\dfrac{x^2-3x}{x^2-4}$

$$=\frac{x(x+2)}{(x+3)(x-3)}\times\frac{x(x-3)}{(x+2)(x-2)}=\boldsymbol{\frac{x^2}{(x+3)(x-2)}}$$

(3) $\dfrac{(x-1)^2}{y^3}\times\dfrac{xy^2+y^2}{x^2-1}$

$$=\frac{(x-1)^2}{y^3}\times\frac{y^2(x+1)}{(x+1)(x-1)}=\boldsymbol{\frac{x-1}{y}}$$

(4) $\dfrac{(a+b)^2}{a-b}\div\dfrac{b+a}{(b-a)^2}$

$$=\frac{(a+b)^2}{a-b}\times\frac{(a-b)^2}{a+b}=\boldsymbol{(a+b)(a-b)}$$

J65b

（5） $\dfrac{a-b}{a^2+ab} \div \dfrac{ab-b^2}{a+b}$

$= \dfrac{a-b}{a^2+ab} \times \dfrac{a+b}{ab-b^2} = \dfrac{a-b}{a(a+b)} \times \dfrac{a+b}{b(a-b)} = \boldsymbol{\dfrac{1}{ab}}$

（6） $\dfrac{a(a-b)^2}{b(a^2-b^2)} \times \dfrac{ab+b^2}{a-b}$

$= \dfrac{a(a-b)^2}{b(a+b)(a-b)} \times \dfrac{b(a+b)}{a-b} = \boldsymbol{a}$

（7） $\dfrac{x+1}{x^2+4x+4} \times \dfrac{x+2}{x^2+6x+9} \div \dfrac{(x+1)^2}{x+3}$

$= \dfrac{x+1}{(x+2)^2} \times \dfrac{x+2}{(x+3)^2} \times \dfrac{x+3}{(x+1)^2} = \boldsymbol{\dfrac{1}{(x+1)(x+2)(x+3)}}$

（8） $\dfrac{x+a}{(x-a)^2} \times \dfrac{x^2-a^2}{x^2+a^2} \times \dfrac{x^4-a^4}{(x+a)^3}$

$= \dfrac{x+a}{(x-a)^2} \times \dfrac{(x+a)(x-a)}{x^2+a^2} \times \dfrac{(x^2+a^2)(x^2-a^2)}{(x+a)^3}$

$= \dfrac{x+a}{(x-a)^2} \times \dfrac{(x+a)(x-a)}{x^2+a^2} \times \dfrac{(x^2+a^2)(x+a)(x-a)}{(x+a)^3} = \mathbf{1}$

J 66a KUMON

Fractional Expressions

Simplify the following fractional expressions.

Ex.

$$\frac{1}{2x}+\frac{1}{3x}=\frac{3}{6x}+\frac{2}{6x}=\frac{5}{6x}$$

☜ First find the common denominator.

(1) $\frac{1}{4x}+\frac{1}{6x}=\frac{3}{12x}+\frac{2}{12x}=\boldsymbol{\frac{5}{12x}}$

(2) $\frac{1}{2x}-\frac{1}{3x}=\frac{3}{6x}-\frac{2}{6x}=\boldsymbol{\frac{1}{6x}}$

(3) $\frac{1}{2x}-\frac{3}{4x}+\frac{5}{6x}=\frac{6}{12x}-\frac{9}{12x}+\frac{10}{12x}=\boldsymbol{\frac{7}{12x}}$

(4) $\frac{2b}{3a}-\frac{3b}{4a}+\frac{5b}{6a}=\frac{8b}{12a}-\frac{9b}{12a}+\frac{10b}{12a}=\frac{9b}{12a}=\boldsymbol{\frac{3b}{4a}}$

(5) $\frac{2c}{3ab}+\frac{3c}{4ab}+\frac{c}{2ab}=\frac{8c}{12ab}+\frac{9c}{12ab}+\frac{6c}{12ab}=\boldsymbol{\frac{23c}{12ab}}$

J66b

Ex.

$$\frac{1}{a}+\frac{1}{b}=\frac{b}{ab}+\frac{a}{ab}=\frac{a+b}{ab}$$

(6) $\dfrac{3}{ax}+\dfrac{1}{bx}=\dfrac{3b}{abx}+\dfrac{a}{abx}=\boldsymbol{\dfrac{a+3b}{abx}}$

(7) $\dfrac{5}{x}+\dfrac{3x+1}{x^2}=\dfrac{5x}{x^2}+\dfrac{3x+1}{x^2}=\boldsymbol{\dfrac{8x+1}{x^2}}$

(8) $\dfrac{3}{x}-\dfrac{3x-1}{x^2}=\dfrac{3x}{x^2}-\dfrac{3x-1}{x^2}=\boldsymbol{\dfrac{1}{x^2}}$

(9) $\dfrac{2}{x}+\dfrac{3}{x+3}=\dfrac{2(x+3)}{x(x+3)}+\dfrac{3x}{x(x+3)}=\boldsymbol{\dfrac{5x+6}{x(x+3)}}$

(10) $\dfrac{2}{x-4}+\dfrac{2}{x+4}=\dfrac{2(x+4)}{(x-4)(x+4)}+\dfrac{2(x-4)}{(x-4)(x+4)}=\boldsymbol{\dfrac{4x}{(x-4)(x+4)}}$

J 67a KUMON

Fractional Expressions

Simplify the following fractional expressions.

Ex.

$$\frac{x-5}{x+5}+\frac{x+5}{x-5}=\frac{(x-5)^2}{(x+5)(x-5)}+\frac{(x+5)^2}{(x+5)(x-5)}=\frac{2(x^2+25)}{(x+5)(x-5)}$$

(1) $\frac{x-5}{x+5}-\frac{x+5}{x-5}=\frac{(x-5)^2}{(x+5)(x-5)}-\frac{(x+5)^2}{(x+5)(x-5)}$

$$=\frac{x^2-10x+25-x^2-10x-25}{(x+5)(x-5)}=-\boldsymbol{\frac{20x}{(x+5)(x-5)}}$$

(2) $\frac{x+1}{x^2}+\frac{2}{x}$

$$=\frac{x+1}{x^2}+\frac{2x}{x^2}=\boldsymbol{\frac{3x+1}{x^2}}$$

(3) $\frac{x+1}{(x+2)^2}-\frac{1}{x+2}$

$$=\frac{x+1}{(x+2)^2}-\frac{x+2}{(x+2)^2}=-\boldsymbol{\frac{1}{(x+2)^2}}$$

(4) $\frac{3}{ab}+\frac{2}{bc}$

$$=\frac{3c}{abc}+\frac{2a}{abc}=\boldsymbol{\frac{2a+3c}{abc}}$$

(5) $\frac{3}{(x+1)(x+2)}+\frac{2}{(x+2)(x+3)}$

$$=\frac{3(x+3)}{(x+1)(x+2)(x+3)}+\frac{2(x+1)}{(x+1)(x+2)(x+3)}=\boldsymbol{\frac{5x+11}{(x+1)(x+2)(x+3)}}$$

J 67b

Ex.

$$x+\frac{x}{x-1}=\frac{x(x-1)}{x-1}+\frac{x}{x-1}=\frac{x^2}{x-1}$$

(6) $x+1+\dfrac{1}{x-1}$

$$=\frac{(x+1)(x-1)}{x-1}+\frac{1}{x-1}=\frac{x^2-1}{x-1}+\frac{1}{x-1}=\boldsymbol{\frac{x^2}{x-1}}$$

(7) $1-\dfrac{4}{x-1}+\dfrac{12}{x-3}$

$$=\frac{(x-1)(x-3)}{(x-1)(x-3)}-\frac{4(x-3)}{(x-1)(x-3)}+\frac{12(x-1)}{(x-1)(x-3)}$$

$$=\frac{x^2-4x+3-4x+12+12x-12}{(x-1)(x-3)}=\boldsymbol{\frac{x^2+4x+3}{(x-1)(x-3)}}$$

$$\left[=\boldsymbol{\frac{(x+3)(x+1)}{(x-1)(x-3)}}\right]$$

(8) $1-\dfrac{a}{1+a}+\dfrac{a}{1-a}$

$$=\frac{(1+a)(1-a)}{(1+a)(1-a)}-\frac{a(1-a)}{(1+a)(1-a)}+\frac{a(1+a)}{(1+a)(1-a)}$$

$$=\frac{1-a^2-a+a^2+a+a^2}{(1+a)(1-a)}$$

$$=\boldsymbol{\frac{a^2+1}{(1+a)(1-a)}}\quad\left[=\boldsymbol{-\frac{a^2+1}{(a+1)(a-1)}}\right]$$

(9)* $\dfrac{1}{a-b}+\dfrac{1}{b-a}=\dfrac{1}{a-b}-\dfrac{1}{a-b}=\mathbf{0}$

J 68a KUMON

Fractional Expressions

Simplify the following fractional expressions.

Ex.

$$\frac{1}{x+7}-\frac{1}{x-1}+\frac{1}{x-7}-\frac{1}{x+1}$$

Rearrange terms into matching pairs.

$$=\frac{(x-7)+(x+7)}{(x+7)(x-7)}-\frac{(x+1)+(x-1)}{(x-1)(x+1)}$$

$$=\frac{2x}{(x+7)(x-7)}-\frac{2x}{(x-1)(x+1)}$$

$$=\frac{2x[(x-1)(x+1)-(x+7)(x-7)]}{(x+7)(x-7)(x-1)(x+1)}$$

$$=\frac{2x[(x^2-1)-(x^2-49)]}{(x+7)(x-7)(x-1)(x+1)}=\frac{96x}{(x+7)(x-7)(x-1)(x+1)}$$

(1) $\frac{1}{x-3}-\frac{1}{x+3}-\frac{1}{x-1}+\frac{1}{x+1}$

$$=\frac{(x+3)-(x-3)}{(x-3)(x+3)}-\frac{(x+1)-(x-1)}{(x-1)(x+1)}$$

$$=\frac{6}{(x-3)(x+3)}-\frac{2}{(x-1)(x+1)}$$

$$=\frac{6(x-1)(x+1)-2(x-3)(x+3)}{(x-3)(x+3)(x-1)(x+1)}$$

$$=\frac{6(x^2-1)-2(x^2-9)}{(x-3)(x+3)(x-1)(x+1)}$$

$$=\boldsymbol{\frac{4x^2+12}{(x-3)(x+3)(x-1)(x+1)}}$$

$$\left[=\boldsymbol{\frac{4(x^2+3)}{(x-3)(x+3)(x-1)(x+1)}}\right]$$

J68b

(2) $\dfrac{1}{a+1}-\dfrac{1}{a+3}-\dfrac{1}{a+2}+\dfrac{1}{a+4}$

$$=\frac{(a+2)-(a+1)}{(a+1)(a+2)}-\frac{(a+4)-(a+3)}{(a+3)(a+4)}$$

$$=\frac{1}{(a+1)(a+2)}-\frac{1}{(a+3)(a+4)}$$

$$=\frac{(a+3)(a+4)-(a+1)(a+2)}{(a+1)(a+2)(a+3)(a+4)}$$

$$=\frac{(a^2+7a+12)-(a^2+3a+2)}{(a+1)(a+2)(a+3)(a+4)}$$

$$=\boldsymbol{\frac{4a+10}{(a+1)(a+2)(a+3)(a+4)}}$$

$$\left[=\boldsymbol{\frac{2(2a+5)}{(a+1)(a+2)(a+3)(a+4)}}\right]$$

(3) $\dfrac{1}{a-1}-\dfrac{1}{a+1}-\dfrac{2}{a^2+1}-\dfrac{4}{a^4+1}$

First find the common denominator of $\dfrac{1}{a-1}$ and $\dfrac{1}{a+1}$.

$$=\frac{(a+1)-(a-1)}{(a-1)(a+1)}-\frac{2}{a^2+1}-\frac{4}{a^4+1}$$

$$=\frac{2}{a^2-1}-\frac{2}{a^2+1}-\frac{4}{a^4+1}$$

$$=\frac{2(a^2+1)-2(a^2-1)}{(a^2-1)(a^2+1)}-\frac{4}{a^4+1}$$

$$=\frac{4}{a^4-1}-\frac{4}{a^4+1}$$

$$=\frac{4(a^4+1)-4(a^4-1)}{(a^4-1)(a^4+1)}$$

$$=\boldsymbol{\frac{8}{(a^4-1)(a^4+1)}}$$

$$\left[=\boldsymbol{\frac{8}{a^8-1}}\right]$$

J69a KUMON

Fractional Expressions

Simplify the following fractional expressions.

Ex.

$$\frac{\frac{3}{5}}{6}=\frac{3}{5}\div 6=\frac{1}{10}$$

(1) $$\frac{\frac{3}{4}}{9}=\frac{3}{4}\div 9=\frac{3}{4}\times\frac{1}{9}=\mathbf{\frac{1}{12}}$$

(2) $$\frac{3}{\frac{5}{6}}=3\div\frac{5}{6}=3\times\frac{6}{5}=\mathbf{\frac{18}{5}}$$

Ex.

$$\frac{1+\frac{1}{a}}{1-\frac{1}{a}}=\left(1+\frac{1}{a}\right)\div\left(1-\frac{1}{a}\right)=\frac{a+1}{a}\times\frac{a}{a-1}=\frac{a+1}{a-1}$$

(3) $$\frac{1}{1+\frac{1}{x}}=1\div\left(1+\frac{1}{x}\right)=1\times\frac{x}{x+1}=\boldsymbol{\frac{x}{x+1}}$$

(4) $$\frac{1+\frac{1}{x}}{x+1}=\left(1+\frac{1}{x}\right)\div(x+1)=\frac{x+1}{x}\times\frac{1}{x+1}=\boldsymbol{\frac{1}{x}}$$

J69b

Ex.

$$\frac{1+\dfrac{1}{a}}{1-\dfrac{1}{a}}=\frac{\left(1+\dfrac{1}{a}\right)a}{\left(1-\dfrac{1}{a}\right)a}=\frac{a+1}{a-1}$$

☜ Multiply numerator and denominator by a.

(5) $$\frac{1}{1+\dfrac{1}{x}}=\frac{1\cdot x}{\left(1+\dfrac{1}{x}\right)x}=\boldsymbol{\frac{x}{x+1}}$$

(6) $$\frac{1}{1+\dfrac{1}{x+2}}=\frac{1\cdot(x+2)}{\left(1+\dfrac{1}{x+2}\right)(x+2)}=\boldsymbol{\frac{x+2}{x+3}}$$

(7) $$\frac{1}{1-\dfrac{1}{1+x}}=\frac{1+x}{\left(1-\dfrac{1}{1+x}\right)(1+x)}=\boldsymbol{\frac{x+1}{x}}$$

(8) $$\frac{1-\dfrac{2}{x-5}}{1+\dfrac{3}{x-5}}=\frac{\left(1-\dfrac{2}{x-5}\right)(x-5)}{\left(1+\dfrac{3}{x-5}\right)(x-5)}=\boldsymbol{\frac{x-7}{x-2}}$$

J 70a KUMON

Fractional Expressions

Simplify the following fractional expressions.

(1) $\dfrac{x^2+5x+6}{x^2+4x+4}$

$$=\frac{(x+3)(x+2)}{(x+2)^2}$$

$$=\boldsymbol{\frac{x+3}{x+2}}$$

(2) $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$

$$=\frac{(a+b+c+d)(a+b-c-d)}{(a+c+b+d)(a+c-b-d)}$$

$$=\boldsymbol{\frac{a+b-c-d}{a-b+c-d}}$$

(3) $\dfrac{12a^2-6a}{2a^2-a-1}\times\dfrac{4a^2-1}{12a^3+18a^2-12a}$

$$=\frac{6a(2a-1)}{(2a+1)(a-1)}\times\frac{(2a+1)(2a-1)}{6a(2a^2+3a-2)}$$

$$=\frac{6a(2a-1)}{(2a+1)(a-1)}\times\frac{(2a+1)(2a-1)}{6a(2a-1)(a+2)}$$

$$=\boldsymbol{\frac{2a-1}{(a-1)(a+2)}}$$

J70b

(4) $\dfrac{a-b}{ab}+\dfrac{b-c}{bc}+\dfrac{c-a}{ca}$

$=\dfrac{c(a-b)+a(b-c)+b(c-a)}{abc}$

$=\dfrac{ac-bc+ab-ac+bc-ab}{abc}$

$=\mathbf{0}$

(5) $\dfrac{2}{x-2}-\dfrac{1}{x+2}+\dfrac{6-x}{x^2-4}$

$=\dfrac{2(x+2)-(x-2)}{(x-2)(x+2)}+\dfrac{6-x}{x^2-4}$

$=\dfrac{x+6+6-x}{x^2-4}$

$=\boldsymbol{\dfrac{12}{x^2-4}}$

$\left[=\boldsymbol{\dfrac{12}{(x+2)(x-2)}}\right]$

(6) $\dfrac{3}{(x+1)(x+2)}-\dfrac{3}{(x+2)(x+3)}$

$=\dfrac{3(x+3)-3(x+1)}{(x+1)(x+2)(x+3)}$

$=\dfrac{3x+9-3x-3}{(x+1)(x+2)(x+3)}$

$=\boldsymbol{\dfrac{6}{(x+1)(x+2)(x+3)}}$

J71a KUMON

Irrational Numbers I

Expand the following expressions.

Ex.

$\sqrt{3}(2\sqrt{6}+\sqrt{3})=2\sqrt{18}+3=6\sqrt{2}+3$

$(2\sqrt{3}+\sqrt{2})(\sqrt{3}+3\sqrt{2})=6+6\sqrt{6}+\sqrt{6}+6=12+7\sqrt{6}$

(1) $\sqrt{3}(2\sqrt{6}-\sqrt{12})=2\sqrt{18}-\sqrt{36}=\mathbf{6\sqrt{2}-6}$

(2) $(3\sqrt{18}-5\sqrt{8})\sqrt{3}=3\sqrt{54}-5\sqrt{24}=9\sqrt{6}-10\sqrt{6}=\mathbf{-\sqrt{6}}$

(3) $(\sqrt{5}+5\sqrt{2})(2\sqrt{5}-\sqrt{2})=10+9\sqrt{10}-10=\mathbf{9\sqrt{10}}$

(4) $(\sqrt{5}+4\sqrt{2})(2\sqrt{5}-2\sqrt{2})=10+6\sqrt{10}-16=\mathbf{-6+6\sqrt{10}}$

(5) $(3\sqrt{2}+\sqrt{3})(\sqrt{2}-2\sqrt{3})=6-5\sqrt{6}-6=\mathbf{-5\sqrt{6}}$

Definitions:

Real Numbers

- ***Rational numbers*** :All numbers that can be expressed as fractions or integers, e.g. 4, 0, -3, $\frac{3}{10}$, $-\frac{18}{5}$, $\frac{23}{6}$.
- ***Irrational numbers*** :Numbers that cannot be expressed as fractions, e.g. $\sqrt{2}$, $-\sqrt{5}$, π.

J71b

Ex.

$(\sqrt{6}+\sqrt{3})^2=(\sqrt{6})^2+2\sqrt{6}\times\sqrt{3}+(\sqrt{3})^2=9+2\sqrt{18}=9+6\sqrt{2}$

(6) $(\sqrt{2}+\sqrt{6})^2=2+2\sqrt{12}+6=\mathbf{8+4\sqrt{3}}$

(7) $(\sqrt{6}-\sqrt{3})^2=6-2\sqrt{18}+3=\mathbf{9-6\sqrt{2}}$

(8) $(3\sqrt{2}-\sqrt{6})^2=18-6\sqrt{12}+6=\mathbf{24-12\sqrt{3}}$

(9) $(2\sqrt{3}+\sqrt{6})^2=12+4\sqrt{18}+6=\mathbf{18+12\sqrt{2}}$

(10) $(2\sqrt{6}-3\sqrt{2})^2=24-12\sqrt{12}+18=\mathbf{42-24\sqrt{3}}$

J 72a KUMON

Irrational Numbers I

Expand the following expressions.

Ex.
$(\sqrt{7}+2\sqrt{3})(\sqrt{7}-2\sqrt{3})=(\sqrt{7})^2-(2\sqrt{3})^2=7-12=-5$

(1) $(\sqrt{7}+5)(\sqrt{7}-5)$
$=7-25=\mathbf{-18}$

(2) $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$
$=5-3=\mathbf{2}$

(3) $(3\sqrt{7}+3\sqrt{5})(3\sqrt{7}-3\sqrt{5})$
$=63-45=\mathbf{18}$

(4) $(3\sqrt{2}+2\sqrt{3})(2\sqrt{3}-3\sqrt{2})$
$=(2\sqrt{3}+3\sqrt{2})(2\sqrt{3}-3\sqrt{2})$
$=12-18=\mathbf{-6}$

(5) $(5-\sqrt{5})(-\sqrt{5}-5)$
$=-(5-\sqrt{5})(5+\sqrt{5})$
$=-(25-5)=\mathbf{-20}$

J72b

(6) $(\sqrt{10}-\sqrt{2})^2$

$=10-2\sqrt{20}+2$

$=\mathbf{12-4\sqrt{5}}$

(7) $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3}+1)$

$=2-\sqrt{6}+\sqrt{2}+\sqrt{6}-3+\sqrt{3}$

$=\mathbf{-1+\sqrt{2}+\sqrt{3}}$

(8) $(\sqrt{3}+\sqrt{2}+\sqrt{6})^2$

$=3+2+6+2\sqrt{6}+2\sqrt{12}+2\sqrt{18}$

$=\mathbf{11+2\sqrt{6}+4\sqrt{3}+6\sqrt{2}}$

Use the formula
$(a+b+c)^2$
$=a^2+b^2+c^2+2ab+2bc+2ca$
[J4b]

(9) $(\sqrt{3}-\sqrt{2}+\sqrt{6})^2$

$=3+2+6-2\sqrt{6}-2\sqrt{12}+2\sqrt{18}$

$=\mathbf{11-2\sqrt{6}-4\sqrt{3}+6\sqrt{2}}$

J73a KUMON

Irrational Numbers I

Expand the following expressions.

Ex.

$$(\sqrt{3}+\sqrt{2}+\sqrt{5})(\sqrt{3}+\sqrt{2}-\sqrt{5})$$
$$=[(\sqrt{3}+\sqrt{2})+\sqrt{5}][(\sqrt{3}+\sqrt{2})-\sqrt{5}]$$
$$=(\sqrt{3}+\sqrt{2})^2-5$$
$$=3+2\sqrt{6}+2-5$$
$$=2\sqrt{6}$$

(1) $(\sqrt{3}-\sqrt{2}-\sqrt{5})(\sqrt{3}-\sqrt{2}+\sqrt{5})$
$$=[(\sqrt{3}-\sqrt{2})-\sqrt{5}][(\sqrt{3}-\sqrt{2})+\sqrt{5}]$$
$$=(\sqrt{3}-\sqrt{2})^2-5$$
$$=3-2\sqrt{6}+2-5$$
$$=\mathbf{-2\sqrt{6}}$$

(2) $(\sqrt{3}+\sqrt{2}-\sqrt{5})(\sqrt{3}-\sqrt{2}+\sqrt{5})$
$$=[\sqrt{3}+(\sqrt{2}-\sqrt{5})][\sqrt{3}-(\sqrt{2}-\sqrt{5})]$$
$$=3-(\sqrt{2}-\sqrt{5})^2$$
$$=3-(2-2\sqrt{10}+5)$$
$$=3-2+2\sqrt{10}-5$$
$$=\mathbf{-4+2\sqrt{10}}$$

(3) $(\sqrt{2}-\sqrt{3}+\sqrt{6})(\sqrt{2}+\sqrt{3}-\sqrt{6})$
$$=[\sqrt{2}-(\sqrt{3}-\sqrt{6})][\sqrt{2}+(\sqrt{3}-\sqrt{6})]$$
$$=2-(\sqrt{3}-\sqrt{6})^2$$
$$=2-(3-2\sqrt{18}+6)$$
$$=2-3+2\sqrt{18}-6$$
$$=\mathbf{-7+6\sqrt{2}}$$

J73b

（４）　$(\sqrt{2}-\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3}-\sqrt{5})$

$=[(\sqrt{2}-\sqrt{3})+\sqrt{5}][(\sqrt{2}-\sqrt{3})-\sqrt{5}]$

$=(\sqrt{2}-\sqrt{3})^2-5$

$=2-2\sqrt{6}+3-5$

$=\mathbf{-2\sqrt{6}}$

（５）　$(\sqrt{2}-\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5})$

$=[\sqrt{2}-(\sqrt{3}-\sqrt{5})][\sqrt{2}+(\sqrt{3}-\sqrt{5})]$

$=2-(\sqrt{3}-\sqrt{5})^2$

$=2-(3-2\sqrt{15}+5)$

$=2-3+2\sqrt{15}-5$

$=\mathbf{-6+2\sqrt{15}}$

（６）　$(\sqrt{3}+\sqrt{5}+\sqrt{7}+1)(\sqrt{3}+\sqrt{5}-\sqrt{7}-1)$

$=[(\sqrt{3}+\sqrt{5})+(\sqrt{7}+1)][(\sqrt{3}+\sqrt{5})-(\sqrt{7}+1)]$

$=(\sqrt{3}+\sqrt{5})^2-(\sqrt{7}+1)^2$

$=(3+2\sqrt{15}+5)-(7+2\sqrt{7}+1)$

$=3+2\sqrt{15}+5-7-2\sqrt{7}-1$

$=\mathbf{2\sqrt{15}-2\sqrt{7}}$

J74a KUMON

Irrational Numbers I

1. Rationalise the denominators.
   (Eliminate the radical sign $\sqrt{}$ from the denominator.)

Ex.

$$\frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}} = \frac{\sqrt{15}}{5}$$

(1) $\dfrac{\sqrt{5}}{\sqrt{7}} = \dfrac{\sqrt{5}\times\sqrt{7}}{\sqrt{7}\times\sqrt{7}} = \boldsymbol{\dfrac{\sqrt{35}}{7}}$

(2) $\sqrt{\dfrac{4}{5}} = \dfrac{\sqrt{4}}{\sqrt{5}} = \dfrac{2}{\sqrt{5}} = \dfrac{2\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}} = \boldsymbol{\dfrac{2\sqrt{5}}{5}}$

(3) $\dfrac{\sqrt{2}}{2\sqrt{3}} = \dfrac{\sqrt{2}\times\sqrt{3}}{2\sqrt{3}\times\sqrt{3}} = \boldsymbol{\dfrac{\sqrt{6}}{6}}$

(4) $\dfrac{\sqrt{6}}{3\sqrt{2}} = \dfrac{\sqrt{6}\times\sqrt{2}}{3\sqrt{2}\times\sqrt{2}} = \dfrac{2\sqrt{3}}{6} = \boldsymbol{\dfrac{\sqrt{3}}{3}}$

(5) $\sqrt{\dfrac{1}{20}} = \dfrac{\sqrt{1}}{\sqrt{20}} = \dfrac{1}{\sqrt{20}} = \dfrac{1}{2\sqrt{5}} = \dfrac{1\times\sqrt{5}}{2\sqrt{5}\times\sqrt{5}} = \boldsymbol{\dfrac{\sqrt{5}}{10}}$

(6) $\dfrac{5\sqrt{2}}{2\sqrt{6}} = \dfrac{5\sqrt{2}\times\sqrt{6}}{2\sqrt{6}\times\sqrt{6}} = \dfrac{5\sqrt{12}}{12} = \dfrac{10\sqrt{3}}{12} = \boldsymbol{\dfrac{5\sqrt{3}}{6}}$

J74b

2. Rationalise the denominators, then evaluate.

Ex.

$$\sqrt{3}+\frac{2}{\sqrt{3}}=\sqrt{3}+\frac{2\sqrt{3}}{3}=\frac{5\sqrt{3}}{3}$$

(1) $\dfrac{5}{2\sqrt{3}}-\sqrt{3}=\dfrac{5\sqrt{3}}{6}-\sqrt{3}=\boldsymbol{-\dfrac{\sqrt{3}}{6}}$

(2) $\dfrac{2\sqrt{3}}{\sqrt{2}}+2\sqrt{6}=\dfrac{2\sqrt{6}}{2}+2\sqrt{6}=\boldsymbol{3\sqrt{6}}$

(3) $\dfrac{3\sqrt{8}}{2\sqrt{3}}+\dfrac{\sqrt{6}}{3}=\dfrac{3\sqrt{24}}{6}+\dfrac{\sqrt{6}}{3}=\sqrt{6}+\dfrac{\sqrt{6}}{3}=\boldsymbol{\dfrac{4\sqrt{6}}{3}}$

(4) $3\sqrt{50}-4\sqrt{32}+\sqrt{\dfrac{1}{2}}=15\sqrt{2}-16\sqrt{2}+\dfrac{\sqrt{2}}{2}=\boldsymbol{-\dfrac{\sqrt{2}}{2}}$

(5) $\dfrac{6-2\sqrt{3}}{\sqrt{3}}=\dfrac{\sqrt{3}(6-2\sqrt{3})}{3}=\dfrac{6\sqrt{3}-6}{3}=\boldsymbol{2\sqrt{3}-2}$

(6) $\sqrt{\dfrac{8}{3}}-\sqrt{\dfrac{3}{8}}=\dfrac{\sqrt{24}}{3}-\dfrac{\sqrt{24}}{8}=\dfrac{2\sqrt{6}}{3}-\dfrac{\sqrt{6}}{4}=\dfrac{8\sqrt{6}}{12}-\dfrac{3\sqrt{6}}{12}=\boldsymbol{\dfrac{5\sqrt{6}}{12}}$

J 75a KUMON

Irrational Numbers I

Rationalise the denominators.

Ex.

$$\frac{1}{\sqrt{3}+1}=\frac{\sqrt{3}-1}{(\sqrt{3}+1)(\sqrt{3}-1)}=\frac{\sqrt{3}-1}{2}$$

(1) $\dfrac{1}{\sqrt{5}+1}=\dfrac{\sqrt{5}-1}{(\sqrt{5}+1)(\sqrt{5}-1)}=\boldsymbol{\dfrac{\sqrt{5}-1}{4}}$

(2) $\dfrac{4}{\sqrt{3}+1}=\dfrac{4(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}=\dfrac{4(\sqrt{3}-1)}{2}=\boldsymbol{2\sqrt{3}-2}$

(3) $\dfrac{4}{\sqrt{5}+1}=\dfrac{4(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)}=\dfrac{4(\sqrt{5}-1)}{4}=\boldsymbol{\sqrt{5}-1}$

(4) $\dfrac{1}{\sqrt{3}+\sqrt{2}}=\dfrac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}=\dfrac{\sqrt{3}-\sqrt{2}}{1}=\boldsymbol{\sqrt{3}-\sqrt{2}}$

(5) $\dfrac{1}{\sqrt{3}-\sqrt{2}}=\dfrac{\sqrt{3}+\sqrt{2}}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\dfrac{\sqrt{3}+\sqrt{2}}{1}=\boldsymbol{\sqrt{3}+\sqrt{2}}$

J75b

Ex.

$$\frac{\sqrt{2}+1}{\sqrt{2}-1}=\frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)}=3+2\sqrt{2}$$

(6) $$\frac{\sqrt{2}-1}{\sqrt{2}+1}=\frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}=\frac{2-2\sqrt{2}+1}{1}=\mathbf{3-2\sqrt{2}}$$

(7) $$\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}=\frac{(\sqrt{5}+\sqrt{3})^2}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})}=\frac{8+2\sqrt{15}}{2}=\mathbf{4+\sqrt{15}}$$

(8) $$\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{(\sqrt{5}-\sqrt{3})^2}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})}=\frac{8-2\sqrt{15}}{2}=\mathbf{4-\sqrt{15}}$$

(9) $$\frac{\sqrt{5}+2\sqrt{3}}{\sqrt{5}-\sqrt{3}}=\frac{(\sqrt{5}+2\sqrt{3})(\sqrt{5}+\sqrt{3})}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})}=\mathbf{\frac{11+3\sqrt{15}}{2}}$$

(10) $$\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+2\sqrt{3}}=\frac{(\sqrt{5}-\sqrt{3})(\sqrt{5}-2\sqrt{3})}{(\sqrt{5}+2\sqrt{3})(\sqrt{5}-2\sqrt{3})}=\frac{11-3\sqrt{15}}{-7}=\mathbf{-\frac{11-3\sqrt{15}}{7}}$$

J76a KUMON

Irrational Numbers I

1. Rationalise the denominators.

(1) $\dfrac{\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

$$=\frac{\sqrt{2}(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\frac{\sqrt{6}+2}{1}=\boldsymbol{\sqrt{6}+2}$$

(2) $\dfrac{\sqrt{6}-2\sqrt{3}}{\sqrt{6}+2\sqrt{3}}$

$$=\frac{(\sqrt{6}-2\sqrt{3})^2}{(\sqrt{6}+2\sqrt{3})(\sqrt{6}-2\sqrt{3})}=\frac{18-12\sqrt{2}}{-6}=\boldsymbol{-3+2\sqrt{2}}$$

(3) $\dfrac{2\sqrt{3}-4\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

$$=\frac{(2\sqrt{3}-4\sqrt{2})(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\boldsymbol{-2-2\sqrt{6}}$$

(4) $\dfrac{2\sqrt{3}+8}{3+\sqrt{3}}$

$$=\frac{(2\sqrt{3}+8)(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}=\frac{-2\sqrt{3}+18}{6}=\boldsymbol{\frac{-\sqrt{3}+9}{3}}$$

(5) $\dfrac{3\sqrt{2}+2\sqrt{3}}{2\sqrt{3}-3\sqrt{2}}$

$$=\frac{(3\sqrt{2}+2\sqrt{3})(2\sqrt{3}+3\sqrt{2})}{(2\sqrt{3}-3\sqrt{2})(2\sqrt{3}+3\sqrt{2})}=\frac{12\sqrt{6}+30}{-6}=\boldsymbol{-2\sqrt{6}-5}$$

J76b

2. Simplify the following expressions.

(1) $\dfrac{1}{(\sqrt{3}+\sqrt{2})^2}$

$$=\frac{(\sqrt{3}-\sqrt{2})^2}{(\sqrt{3}+\sqrt{2})^2(\sqrt{3}-\sqrt{2})^2}=\frac{5-2\sqrt{6}}{(3-2)^2}=\mathbf{5-2\sqrt{6}}$$

(2) $\dfrac{1}{(\sqrt{10}-\sqrt{2})^2}$

$$=\frac{(\sqrt{10}+\sqrt{2})^2}{(\sqrt{10}-\sqrt{2})^2(\sqrt{10}+\sqrt{2})^2}=\frac{12+4\sqrt{5}}{(10-2)^2}=\frac{12+4\sqrt{5}}{64}=\mathbf{\frac{3+\sqrt{5}}{16}}$$

(3) $\dfrac{3\sqrt{2}}{\sqrt{6}-\sqrt{3}}-\dfrac{3}{3-\sqrt{6}}$

$$=\frac{3\sqrt{2}(\sqrt{6}+\sqrt{3})}{(\sqrt{6}-\sqrt{3})(\sqrt{6}+\sqrt{3})}-\frac{3(3+\sqrt{6})}{(3-\sqrt{6})(3+\sqrt{6})}$$

$$=\frac{6\sqrt{3}+3\sqrt{6}}{3}-\frac{9+3\sqrt{6}}{3}$$

$$=2\sqrt{3}+\sqrt{6}-3-\sqrt{6}=\mathbf{2\sqrt{3}-3}$$

(4) $\dfrac{1}{(\sqrt{3}-\sqrt{2})^2}+\dfrac{1}{(\sqrt{3}+\sqrt{2})^2}$

$$=\frac{(\sqrt{3}+\sqrt{2})^2+(\sqrt{3}-\sqrt{2})^2}{(\sqrt{3}-\sqrt{2})^2(\sqrt{3}+\sqrt{2})^2}=\frac{5+2\sqrt{6}+5-2\sqrt{6}}{(3-2)^2}=\mathbf{10}$$

J77a KUMON

Irrational Numbers I

1. Simplify the following expressions.

(1) $(\sqrt{3}+\sqrt{2})^2 = 3+2\sqrt{6}+2 = \mathbf{5+2\sqrt{6}}$

(2) $(\sqrt{5}+\sqrt{2})^2 = 5+2\sqrt{10}+2 = \mathbf{7+2\sqrt{10}}$

(3) $(\sqrt{7}+\sqrt{2})^2 = 7+2\sqrt{14}+2 = \mathbf{9+2\sqrt{14}}$

(4) $(\sqrt{2}+1)^2 = 2+2\sqrt{2}+1 = \mathbf{3+2\sqrt{2}}$

(5) $(\sqrt{5}-\sqrt{3})^2 = 5-2\sqrt{15}+3 = \mathbf{8-2\sqrt{15}}$

(6) $(\sqrt{5}-\sqrt{2})^2 = 5-2\sqrt{10}+2 = \mathbf{7-2\sqrt{10}}$

J77b

From $(\sqrt{3}+\sqrt{2})^2 = 5+2\sqrt{6}$, we can deduce:

$$\sqrt{5+2\sqrt{6}} = \sqrt{(\sqrt{3}+\sqrt{2})^2} = \sqrt{3}+\sqrt{2} \quad \text{(see Note below)}$$

(A term with a radical sign under a radical sign is called a ***double radical***.)

2. Simplify the following expressions.

Ex.

$$\underset{\substack{\uparrow \\ 5+2}}{\sqrt{7}}\;\underset{\substack{\uparrow \\ 5\times 2}}{+2\sqrt{10}} = \sqrt{5}+\sqrt{2}$$

☜ Look for two numbers that multiply to 10 and add to 7.

(1) $\sqrt{9+2\sqrt{14}} = \boldsymbol{\sqrt{7}+\sqrt{2}}$

(2) $\sqrt{3+2\sqrt{2}} = \sqrt{2}+\sqrt{1} = \boldsymbol{\sqrt{2}+1}$

(3) $\sqrt{6+2\sqrt{5}} = \sqrt{5}+\sqrt{1} = \boldsymbol{\sqrt{5}+1}$

(4) $\sqrt{6+2\sqrt{8}} = \sqrt{4}+\sqrt{2} = \boldsymbol{2+\sqrt{2}}$

(5) $\sqrt{10+2\sqrt{21}} = \boldsymbol{\sqrt{7}+\sqrt{3}}$

Note: $\sqrt{5+2\sqrt{6}} = -\sqrt{3}-\sqrt{2}$ is <u>not correct</u>.

($\sqrt{5+2\sqrt{6}}$ should be positive, but $-\sqrt{3}-\sqrt{2}$ is negative.)

J 78a KUMON

Irrational Numbers I

Simplify the following expressions.

Ex.

$\sqrt{8-2\sqrt{15}}=\sqrt{5}-\sqrt{3}$ (The answer is not $\sqrt{3}-\sqrt{5}$, see Note below.)

$\uparrow$ $5+3$ $\uparrow$ 5×3

☞ Look for two numbers that multiply to 15 and add to 8.

(1) $\sqrt{7-2\sqrt{10}}=\sqrt{\mathbf{5}}-\sqrt{\mathbf{2}}$

(2) $\sqrt{10-2\sqrt{21}}=\sqrt{\mathbf{7}}-\sqrt{\mathbf{3}}$

(3) $\sqrt{15-2\sqrt{50}}=\sqrt{\mathbf{10}}-\sqrt{\mathbf{5}}$

(4) $\sqrt{6-2\sqrt{5}}=\sqrt{5}-\sqrt{1}=\sqrt{\mathbf{5}}-\mathbf{1}$

(5) $\sqrt{5-2\sqrt{6}}=\sqrt{\mathbf{3}}-\sqrt{\mathbf{2}}$

Note: $\sqrt{8-2\sqrt{15}}=\sqrt{3}-\sqrt{5}$ is <u>not correct</u>.

($\sqrt{8-2\sqrt{15}}$ should be positive, but $\sqrt{3}-\sqrt{5}$ is negative.)

J78b

Ex.

$\sqrt{4+\sqrt{12}}=\sqrt{4+2\sqrt{3}}=\sqrt{3}+\sqrt{1}=\sqrt{3}+1$

$\sqrt{7+4\sqrt{3}}=\sqrt{7+2\sqrt{12}}=\sqrt{4}+\sqrt{3}=2+\sqrt{3}$

Rewrite to give 2 in the position shown.

(6) $\sqrt{6-\sqrt{20}}=\sqrt{6-2\sqrt{5}}=\sqrt{5}-\sqrt{1}=\boldsymbol{\sqrt{5}-1}$

(7) $\sqrt{7-\sqrt{48}}=\sqrt{7-2\sqrt{12}}=\sqrt{4}-\sqrt{3}=\boldsymbol{2-\sqrt{3}}$

(8) $\sqrt{8+4\sqrt{3}}=\sqrt{8+2\sqrt{12}}=\boldsymbol{\sqrt{6}+\sqrt{2}}$

(9) $\sqrt{12-6\sqrt{3}}=\sqrt{12-2\sqrt{27}}=\sqrt{9}-\sqrt{3}=\boldsymbol{3-\sqrt{3}}$

(10) $\sqrt{19+8\sqrt{3}}=\sqrt{19+2\sqrt{48}}=\sqrt{16}+\sqrt{3}=\boldsymbol{4+\sqrt{3}}$

J79a

KUMON

Irrational Numbers I

Simplify the following expressions.

Ex.

$$\sqrt{2+\sqrt{3}} = \sqrt{\frac{4+2\sqrt{3}}{2}} = \frac{\sqrt{4+2\sqrt{3}}}{\sqrt{2}} = \frac{\sqrt{3}+1}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{2}$$

Note:

$$\sqrt{2+\sqrt{3}} = \sqrt{\frac{2(2+\sqrt{3})}{2}}$$

(1) $\sqrt{3+\sqrt{5}} = \sqrt{\frac{6+2\sqrt{5}}{2}} = \frac{\sqrt{6+2\sqrt{5}}}{\sqrt{2}} = \frac{\sqrt{5}+1}{\sqrt{2}} = \boldsymbol{\frac{\sqrt{10}+\sqrt{2}}{2}}$

(2) $\sqrt{4-\sqrt{7}} = \sqrt{\frac{8-2\sqrt{7}}{2}} = \frac{\sqrt{8-2\sqrt{7}}}{\sqrt{2}} = \frac{\sqrt{7}-1}{\sqrt{2}} = \boldsymbol{\frac{\sqrt{14}-\sqrt{2}}{2}}$

(3) $\sqrt{5-\sqrt{21}} = \sqrt{\frac{10-2\sqrt{21}}{2}} = \frac{\sqrt{10-2\sqrt{21}}}{\sqrt{2}} = \frac{\sqrt{7}-\sqrt{3}}{\sqrt{2}} = \boldsymbol{\frac{\sqrt{14}-\sqrt{6}}{2}}$

(4) $\sqrt{7+\sqrt{45}} = \sqrt{\frac{14+2\sqrt{45}}{2}} = \frac{\sqrt{14+2\sqrt{45}}}{\sqrt{2}} = \frac{3+\sqrt{5}}{\sqrt{2}} = \boldsymbol{\frac{3\sqrt{2}+\sqrt{10}}{2}}$

J79b

Ex.

$\sqrt{9+3\sqrt{8}}=\sqrt{9+\underline{6}\sqrt{2}}=\sqrt{9+\underline{2}\sqrt{18}}=\sqrt{6}+\sqrt{3}$

Pay close attention to the underlined terms.

(5) $\sqrt{28-5\sqrt{12}}$

$=\sqrt{28-10\sqrt{3}}=\sqrt{28-2\sqrt{75}}=\sqrt{25}-\sqrt{3}=\mathbf{5-\sqrt{3}}$

(6) $\sqrt{14+3\sqrt{20}}$

$=\sqrt{14+6\sqrt{5}}=\sqrt{14+2\sqrt{45}}=\sqrt{9}+\sqrt{5}=\mathbf{3+\sqrt{5}}$

(7) $\sqrt{11+2\sqrt{30}}-\sqrt{11-2\sqrt{30}}$

$=(\sqrt{6}+\sqrt{5})-(\sqrt{6}-\sqrt{5})=\mathbf{2\sqrt{5}}$

(8) $\dfrac{1}{\sqrt{11-4\sqrt{7}}}$

$=\dfrac{1}{\sqrt{11-2\sqrt{28}}}=\dfrac{1}{\sqrt{7}-\sqrt{4}}=\dfrac{1}{\sqrt{7}-2}$

$=\dfrac{\sqrt{7}+2}{(\sqrt{7}-2)(\sqrt{7}+2)}=\mathbf{\dfrac{\sqrt{7}+2}{3}}$

J 80a KUMON

Irrational Numbers I

1. Simplify the following expressions.

(1) $\sqrt{4+2\sqrt{3}}=\mathbf{\sqrt{3}+1}$

(2) $\sqrt{8-2\sqrt{15}}=\mathbf{\sqrt{5}-\sqrt{3}}$

(3) $\sqrt{2+\sqrt{3}}=\dfrac{\sqrt{4+2\sqrt{3}}}{\sqrt{2}}=\dfrac{\sqrt{3}+1}{\sqrt{2}}=\mathbf{\dfrac{\sqrt{6}+\sqrt{2}}{2}}$

2. Simplify the following expressions.

(1) $\dfrac{1}{(2-\sqrt{3})^2}+\dfrac{1}{(2+\sqrt{3})^2}$

$=\dfrac{(2+\sqrt{3})^2+(2-\sqrt{3})^2}{(2-\sqrt{3})^2(2+\sqrt{3})^2}$

$=\dfrac{7+4\sqrt{3}+7-4\sqrt{3}}{(4-3)^2}$

$=\mathbf{14}$

(2) $\dfrac{1}{\sqrt{12+2\sqrt{35}}}+\dfrac{1}{\sqrt{12-2\sqrt{35}}}$

$=\dfrac{1}{\sqrt{7}+\sqrt{5}}+\dfrac{1}{\sqrt{7}-\sqrt{5}}$

$=\dfrac{(\sqrt{7}-\sqrt{5})+(\sqrt{7}+\sqrt{5})}{(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})}$

$=\dfrac{2\sqrt{7}}{2}=\mathbf{\sqrt{7}}$

J 80b

3. Simplify the following expressions.

(1) $(3\sqrt{2}+2\sqrt{3}+\sqrt{6})(3\sqrt{2}-2\sqrt{3}+\sqrt{6})$

$=[(3\sqrt{2}+\sqrt{6})+2\sqrt{3}][(3\sqrt{2}+\sqrt{6})-2\sqrt{3}]$

$=(3\sqrt{2}+\sqrt{6})^2-(2\sqrt{3})^2$

$=24+6\sqrt{12}-12$

$=\mathbf{12+12\sqrt{3}}$

(2) $\sqrt{5}-\sqrt{\dfrac{5}{4}}+\sqrt{\dfrac{4}{5}}$

$$=\sqrt{5}-\frac{\sqrt{5}}{2}+\frac{2}{\sqrt{5}}=\sqrt{5}-\frac{\sqrt{5}}{2}+\frac{2\sqrt{5}}{5}=\frac{10\sqrt{5}-5\sqrt{5}+4\sqrt{5}}{10}=\mathbf{\frac{9\sqrt{5}}{10}}$$

(3) $\dfrac{3\sqrt{2}}{\sqrt{6}-\sqrt{3}}-\dfrac{3}{3-\sqrt{6}}$

$$=\frac{3\sqrt{2}(\sqrt{6}+\sqrt{3})}{(\sqrt{6}-\sqrt{3})(\sqrt{6}+\sqrt{3})}-\frac{3(3+\sqrt{6})}{(3-\sqrt{6})(3+\sqrt{6})}$$

$$=\frac{6\sqrt{3}+3\sqrt{6}-9-3\sqrt{6}}{3}$$

$$=\frac{6\sqrt{3}-9}{3}$$

$=\mathbf{2\sqrt{3}-3}$

(4) $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$

$$=\frac{3\sqrt{2}(\sqrt{3}-\sqrt{6})}{(\sqrt{3}+\sqrt{6})(\sqrt{3}-\sqrt{6})}-\frac{4\sqrt{3}(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}+\frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$

$$=\frac{3\sqrt{6}-3\sqrt{12}}{3-6}-\frac{4\sqrt{18}-4\sqrt{6}}{6-2}+\frac{\sqrt{18}-\sqrt{12}}{3-2}$$

$$=\frac{3\sqrt{6}-6\sqrt{3}}{-3}-\frac{12\sqrt{2}-4\sqrt{6}}{4}+\frac{3\sqrt{2}-2\sqrt{3}}{1}$$

$=-\sqrt{6}+2\sqrt{3}-3\sqrt{2}+\sqrt{6}+3\sqrt{2}-2\sqrt{3}$

$=\mathbf{0}$

J81a KUMON

Irrational Numbers II

1. If $x=5+\sqrt{3}$ and $y=5-\sqrt{3}$, evaluate the following expressions.

(1) $x+y=(5+\sqrt{3})+(5-\sqrt{3})=\mathbf{10}$

(2) $xy=(5+\sqrt{3})(5-\sqrt{3})=25-3=\mathbf{22}$

2. If $x=\sqrt{5}+\sqrt{2}$ and $y=\sqrt{5}-\sqrt{2}$, evaluate the following expressions.

(1) $x+y=(\sqrt{5}+\sqrt{2})+(\sqrt{5}-\sqrt{2})=\mathbf{2\sqrt{5}}$

(2) $xy=(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})=\mathbf{3}$

3. If $x=\dfrac{3+\sqrt{5}}{4}$ and $y=\dfrac{3-\sqrt{5}}{4}$, evaluate the following expressions.

(1) $x+y=\left(\dfrac{3+\sqrt{5}}{4}\right)+\left(\dfrac{3-\sqrt{5}}{4}\right)=\dfrac{6}{4}=\mathbf{\dfrac{3}{2}}$

(2) $xy=\left(\dfrac{3+\sqrt{5}}{4}\right)\left(\dfrac{3-\sqrt{5}}{4}\right)=\dfrac{4}{16}=\mathbf{\dfrac{1}{4}}$

J81b

4. If $x=\dfrac{1}{\sqrt{3}+1}$ and $y=\dfrac{1}{\sqrt{3}-1}$, evaluate the following expressions.

(1) $$\begin{aligned} x+y &= \left(\frac{1}{\sqrt{3}+1}\right)+\left(\frac{1}{\sqrt{3}-1}\right) \\ &= \frac{\sqrt{3}-1}{2}+\frac{\sqrt{3}+1}{2} \\ &= \boldsymbol{\sqrt{3}} \end{aligned}$$

(2) $$xy=\left(\frac{1}{\sqrt{3}+1}\right)\left(\frac{1}{\sqrt{3}-1}\right)=\boldsymbol{\frac{1}{2}}$$

5. If $x=\dfrac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$ and $y=\dfrac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}$, evaluate the following expressions.

(1) $$\begin{aligned} x+y &= \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}\right)+\left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}\right) \\ &= \frac{(\sqrt{7}-\sqrt{5})^2}{2}+\frac{(\sqrt{7}+\sqrt{5})^2}{2} \\ &= \frac{12-2\sqrt{35}+12+2\sqrt{35}}{2} \\ &= \mathbf{12} \end{aligned}$$

(2) $$xy=\left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}\right)\left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}\right)=\mathbf{1}$$

J 82a KUMON

Irrational Numbers II

1. If $x+y=a$ and $xy=b$, express the following in terms of a and b.

(1) $x^2+y^2=(x+y)^2-2xy=\boldsymbol{a^2-2b}$

(2) $x^4+y^4=(x^2+y^2)^2-2x^2y^2$

Use your answer from (1).

$=(a^2-2b)^2-2b^2$

$=a^4-4a^2b+4b^2-2b^2$

$=\boldsymbol{a^4-4a^2b+2b^2}$

Use your answer from (1).

(3) $(x-y)^2=x^2-2xy+y^2$

$=x^2+y^2-2xy$

$=a^2-2b-2b$

$=\boldsymbol{a^2-4b}$

[Alternative Solution
$(x-y)^2=(x+y)^2-4xy$
$=\boldsymbol{a^2-4b}$]

(4) $x^3+y^3=(x+y)^3-\boxed{\mathbf{3}}\,xy(x+y)$

$=a^3-3ba$

$=\boldsymbol{a^3-3ab}$

J82b

2. If $x=2+\sqrt{3}$ and $y=2-\sqrt{3}$, evaluate the following expressions.

(1) $x+y=(2+\sqrt{3})+(2-\sqrt{3})$
$=\mathbf{4}$

(2) $xy=(2+\sqrt{3})(2-\sqrt{3})$
$=\mathbf{1}$

(3) $x^2+y^2=(x+y)^2-2xy$
$=16-2$
$=\mathbf{14}$

Use your answers from (1) and (2).

(4) $x^4+y^4=(x^2+y^2)^2-2x^2y^2$
$=14^2-2$
$=196-2$
$=\mathbf{194}$

Use your answers from (2) and (3).

(5) $(x-y)^2=x^2-2xy+y^2$
$=x^2+y^2-2xy$
$=14-2$
$=\mathbf{12}$

Use your answers from (2) and (3).

Alternative Solution
$(x-y)^2=[(2+\sqrt{3})-(2-\sqrt{3})]^2$
$=(2\sqrt{3})^2$
$=\mathbf{12}$

(6) $x^3+y^3=(x+y)^3-3xy(x+y)$
$=64-3\times1\times4$
$=64-12$
$=\mathbf{52}$

Use your answers from (1) and (2).

J 83a KUMON

Irrational Numbers II

1. If $x=\sqrt{5}+\sqrt{3}$ and $y=\sqrt{5}-\sqrt{3}$, evaluate the following expressions.

(1) $x+y=(\sqrt{5}+\sqrt{3})+(\sqrt{5}-\sqrt{3})$
$=\mathbf{2\sqrt{5}}$

(2) $xy=(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$
$=5-3$
$=\mathbf{2}$

(3) $x^2+y^2=(x+y)^2-2xy$
$=(2\sqrt{5})^2-2\times2$
$=20-4$
$=\mathbf{16}$

(4) $x^4+y^4=(x^2+y^2)^2-2x^2y^2$
$=16^2-2\times4$
$=256-8$
$=\mathbf{248}$

(5) $(x-y)^2=x^2-2xy+y^2$
$=16-2\times2$
$=\mathbf{12}$

[Alternative Solution
$(x-y)^2=[(\sqrt{5}+\sqrt{3})-(\sqrt{5}-\sqrt{3})]^2$
$=(2\sqrt{3})^2$
$=\mathbf{12}$]

(6) $x^3+y^3=(x+y)^3-3xy(x+y)$
$=(2\sqrt{5})^3-3\times2\times2\sqrt{5}$
$=40\sqrt{5}-12\sqrt{5}$
$=\mathbf{28\sqrt{5}}$

J83b

2. If $x=4+2\sqrt{3}$ and $y=4-2\sqrt{3}$, evaluate the following expressions.

(1) $x+y=(4+2\sqrt{3})+(4-2\sqrt{3})$
$=\mathbf{8}$

(2) $xy=(4+2\sqrt{3})(4-2\sqrt{3})$
$=16-12$
$=\mathbf{4}$

(3) $x^2+y^2=(x+y)^2-2xy$
$=64-8$
$=\mathbf{56}$

(4) $x^3+y^3=(x+y)^3-3xy(x+y)$
$=8^3-3\times4\times8$
$=512-96$
$=\mathbf{416}$

(5) $\dfrac{y}{x}+\dfrac{x}{y}=\dfrac{y^2+x^2}{xy}$
$=\dfrac{56}{4}$
$=\mathbf{14}$

(6) $\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{y^2+x^2}{x^2y^2}$
$=\dfrac{56}{16}$
$=\dfrac{\mathbf{7}}{\mathbf{2}}$

J 84a KUMON

Irrational Numbers II

1. If $x=\dfrac{1}{2+\sqrt{3}}$ and $y=\dfrac{1}{2-\sqrt{3}}$, evaluate the following expressions.

(1) $$\begin{aligned} x+y &= \left(\frac{1}{2+\sqrt{3}}\right)+\left(\frac{1}{2-\sqrt{3}}\right) \\ &= 2-\sqrt{3}+2+\sqrt{3} \\ &= \mathbf{4} \end{aligned}$$

(2) $$xy=\left(\frac{1}{2+\sqrt{3}}\right)\left(\frac{1}{2-\sqrt{3}}\right)=\mathbf{1}$$

(3) $$\begin{aligned} x^2+y^2 &= (x+y)^2-2xy \\ &= 16-2 \\ &= \mathbf{14} \end{aligned}$$

(4) $$\begin{aligned} x^3+y^3 &= (x+y)^3-3xy(x+y) \\ &= 4^3-3\times1\times4 \\ &= 64-12 \\ &= \mathbf{52} \end{aligned}$$

J84b

2. If $x=\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ and $y=\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$, evaluate the following expressions.

(1) $$\begin{aligned} x+y &= \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}\right)+\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right) \\ &= \frac{(\sqrt{5}-\sqrt{3})^2}{2}+\frac{(\sqrt{5}+\sqrt{3})^2}{2} \\ &= \frac{8-2\sqrt{15}+8+2\sqrt{15}}{2} \\ &= \mathbf{8} \end{aligned}$$

(2) $$xy=\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}\right)\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right)$$

$$=\mathbf{1}$$

(3) $$\begin{aligned} (x-y)^2 &= (x+y)^2-4xy \\ &= 64-4 \\ &= \mathbf{60} \end{aligned}$$

(4) $$\begin{aligned} \frac{x}{y^2}+\frac{y}{x^2} &= \frac{x^3+y^3}{x^2y^2} \\ &= \frac{(x+y)^3-3xy(x+y)}{x^2y^2} \\ &= 8^3-3\times1\times8 \\ &= \mathbf{488} \end{aligned}$$

J 85a KUMON

Irrational Numbers II

1. Evaluate the following expressions.

Ex.

If $x+y=\sqrt{2}$ and $x^2+y^2=6$

$$xy=\frac{(x+y)^2-(x^2+y^2)}{2}=\frac{2-6}{2}=-2$$

(1) If $x+y=\sqrt{6}$ and $x^2+y^2=4$

$$xy=\frac{(x+y)^2-(x^2+y^2)}{2}=\frac{6-4}{2}=\mathbf{1}$$

(2) If $x-y=\sqrt{6}$ and $x^2+y^2=4$

$$xy=-\frac{(x-y)^2-(x^2+y^2)}{2}=-\frac{6-4}{2}=\mathbf{-1}$$

(3) If $x-y=\sqrt{3}$ and $x^2+y^2=5$

$$xy=-\frac{(x-y)^2-(x^2+y^2)}{2}=-\frac{3-5}{2}=\mathbf{1}$$

(4) If $x+y=\sqrt{10}$ and $(x-y)^2=2$

$$xy=\frac{(x+y)^2-(x-y)^2}{4}=\frac{10-2}{4}=\mathbf{2}$$

J85b

2. If $x+y=\sqrt{3}$ and $x^2+y^2=5$, evaluate the following expressions.

(1) xy

$$=\frac{(x+y)^2-(x^2+y^2)}{2}=\frac{3-5}{2}=\mathbf{-1}$$

(2) x^3+y^3

$$\begin{aligned}&=(x+y)^3-3xy(x+y)\\&=(\sqrt{3})^3-3\times(-1)\times\sqrt{3}\\&=3\sqrt{3}+3\sqrt{3}\\&=\mathbf{6\sqrt{3}}\end{aligned}$$

(3) $\dfrac{y}{x}+\dfrac{x}{y}$

$$\begin{aligned}&=\frac{y^2+x^2}{xy}\\&=\frac{5}{-1}\\&=\mathbf{-5}\end{aligned}$$

(4) $\dfrac{y}{x^2}+\dfrac{x}{y^2}$

$$\begin{aligned}&=\frac{y^3+x^3}{x^2y^2}\\&=\mathbf{6\sqrt{3}}\end{aligned}$$

J 86a KUMON

Irrational Numbers II

> Any positive number a has two ***square roots***.
> (For example, 9 has two square roots, 3 and -3.)

1. Complete the following exercises.

(1) The square roots of 16 are: $\boxed{\mathbf{4}}$ and $\boxed{\mathbf{-4}}$

(2) The square roots of 25 are: $\boxed{\mathbf{5}}$ and $\boxed{\mathbf{-5}}$

(3) The square roots of 5^2 are: $\boxed{\mathbf{5}}$ and $\boxed{\mathbf{-5}}$

(4) The square roots of $(-5)^2$ are: $\boxed{\mathbf{5}}$ and $\boxed{\mathbf{-5}}$

> The ***square roots*** of a are the positive root, $\sqrt{a}$,
> and the negative root, $-\sqrt{a}$.
> (For example, $\sqrt{9}=3$ and $-\sqrt{9}=-3$.)

(5) $\sqrt{36}=\boxed{\mathbf{6}}$

(6) $-\sqrt{36}=\boxed{\mathbf{-6}}$

(7) $\sqrt{(-5)^2}=\boxed{\mathbf{5}}$

(8) $-\sqrt{(-5)^2}=\boxed{\mathbf{-5}}$

Note: The symbol $\sqrt{}$ refers to the positive square root.
$\sqrt{a}$ is called the principal square root of a.

J86b

2. Determine if the underlined are true or false. If true, write "True". If false, write "False" and the correct answer.

(1) $\sqrt{5^2} = \underline{5 \text{ and } -5}$ False/correct answer is **5**

(2) $\sqrt{(-5)^2} = \underline{-5}$ False/correct answer is **5**

(3) $\sqrt{9} = \underline{\pm 3}$ False/correct answer is **3**

(4) $-\sqrt{16} = \underline{-4}$ True

(5) $\sqrt{(3-1)^2} = \underline{-2}$ False/correct answer is **2**

(6) $\sqrt{(1-3)^2} = \underline{2}$ True

J 87a KUMON

Irrational Numbers II

1. Evaluate the following expressions.

(1) When $a=-2$, $\sqrt{a^2}=\sqrt{(-2)^2}=\mathbf{2}$

(2) When $a=-1$, $\sqrt{a^2}=\sqrt{(-1)^2}=\mathbf{1}$

(3) When $a=0$, $\sqrt{a^2}=\sqrt{0^2}=\mathbf{0}$

(4) When $a=1$, $\sqrt{a^2}=\sqrt{1^2}=\mathbf{1}$

(5) When $a=2$, $\sqrt{a^2}=\sqrt{2^2}=\mathbf{2}$

(6) When $a=3$, $\sqrt{a^2}=\sqrt{3^2}=\mathbf{3}$

(7) When $a=-3$, $\sqrt{a^2}=\sqrt{(-3)^2}=\mathbf{3}$

2. Fill in each blank with either a, $-a$, a^2, or $-a^2$.

(1) When $a \geqslant 0$, (a is 0 or a positive number), $\sqrt{a^2}=\boxed{\boldsymbol{a}}$

(2) When $a<0$, (a is a negative number), $\sqrt{a^2}=\boxed{\boldsymbol{-a}}$

3. Evaluate the following expressions.

(1) When $x = 5$, $\sqrt{(x-3)^2} = \sqrt{(5-3)^2} = \mathbf{2}$

(2) When $x = 4$, $\sqrt{(x-3)^2} = \sqrt{(4-3)^2} = \mathbf{1}$

(3) When $x = 3$, $\sqrt{(x-3)^2} = \sqrt{(3-3)^2} = \mathbf{0}$

(4) When $x = 2$, $\sqrt{(x-3)^2} = \sqrt{(2-3)^2} = \mathbf{1}$

(5) When $x = 1$, $\sqrt{(x-3)^2} = \sqrt{(1-3)^2} = \mathbf{2}$

(6) When $x = 0$, $\sqrt{(x-3)^2} = \sqrt{(0-3)^2} = \mathbf{3}$

(7) When $x = -1$, $\sqrt{(x-3)^2} = \sqrt{(-1-3)^2} = \mathbf{4}$

Note Summary

- When $a \geqslant b$, $\sqrt{(a-b)^2} = a-b$ ☜ When the term in brackets is positive or zero.
- When $a < b$, $\sqrt{(a-b)^2} = -(a-b)$ ☜ When the term in brackets is negative.

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Irrational Numbers II

Rewrite the following square roots.

Ex.

$$\sqrt{(x-3)^2}=\begin{cases} x-3 & (\text{when } x \geqslant 3) \\ -(x-3) & (\text{when } x<3) \end{cases}$$

From $x-3 \geqslant 0$.

From $x-3<0$.

(1) $\sqrt{(x-2)^2}=\begin{cases} \boldsymbol{x-2} & (\textbf{when } \boldsymbol{x \geqslant 2}) \\ \boldsymbol{-(x-2)} & (\textbf{when } \boldsymbol{x<2}) \end{cases}$

(2) $\sqrt{(x-1)^2}=\begin{cases} \boldsymbol{x-1} & (\textbf{when } \boldsymbol{x \geqslant 1}) \\ \boldsymbol{-(x-1)} & (\textbf{when } \boldsymbol{x<1}) \end{cases}$

(3) $\sqrt{(x+1)^2}=\begin{cases} \boldsymbol{x+1} & (\textbf{when } \boldsymbol{x \geqslant -1}) \\ \boldsymbol{-(x+1)} & (\textbf{when } \boldsymbol{x<-1}) \end{cases}$

(4) $\sqrt{x^2}=\begin{cases} \boldsymbol{x} & (\textbf{when } \boldsymbol{x \geqslant 0}) \\ \boldsymbol{-x} & (\textbf{when } \boldsymbol{x<0}) \end{cases}$

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(5) $\sqrt{(x-4)^2} = \begin{cases} \boldsymbol{x-4} & \textbf{(when } \boldsymbol{x \geqq 4}\textbf{)} \\ \boldsymbol{-(x-4)} & \textbf{(when } \boldsymbol{x < 4}\textbf{)} \end{cases}$

(6) $\sqrt{(x-a)^2} = \begin{cases} \boldsymbol{x-a} & \textbf{(when } \boldsymbol{x \geqq a}\textbf{)} \\ \boldsymbol{-(x-a)} & \textbf{(when } \boldsymbol{x < a}\textbf{)} \end{cases}$

(7) $\sqrt{(x+3)^2} = \begin{cases} \boldsymbol{x+3} & \textbf{(when } \boldsymbol{x \geqq -3}\textbf{)} \\ \boldsymbol{-(x+3)} & \textbf{(when } \boldsymbol{x < -3}\textbf{)} \end{cases}$

(8) $\sqrt{(x+a)^2} = \begin{cases} \boldsymbol{x+a} & \textbf{(when } \boldsymbol{x \geqq -a}\textbf{)} \\ \boldsymbol{-(x+a)} & \textbf{(when } \boldsymbol{x < -a}\textbf{)} \end{cases}$

(9) $\sqrt{(x-2a)^2} = \begin{cases} \boldsymbol{x-2a} & \textbf{(when } \boldsymbol{x \geqq 2a}\textbf{)} \\ \boldsymbol{-(x-2a)} & \textbf{(when } \boldsymbol{x < 2a}\textbf{)} \end{cases}$

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Irrational Numbers II

Ex.

Rewrite the following square roots when $x > 3$.

$\sqrt{(x-1)^2} = x-1$ If $x > 3$, then $x-1 > 0$ is always true.

1. Rewrite the following square roots when $x > 5$.

(1) $\sqrt{(x-2)^2} = \boldsymbol{x-2}$

(2) $\sqrt{(x-4)^2} = \boldsymbol{x-4}$

Ex.

Rewrite the following square roots when $x > 3$.

$$\sqrt{(x-5)^2} = \begin{cases} x-5 & (\text{when } x \geqq 5) \\ -(x-5) & (\text{when } 3 < x < 5) \end{cases}$$

From $x-5 \geqq 0$.

From $x-5 < 0$ and $x > 3$.

2. Rewrite the following square roots when $x > 2$.

(1) $\sqrt{(x-3)^2} = \begin{cases} \boldsymbol{x-3} & (\textbf{when } \boldsymbol{x \geqq 3}) \\ \boldsymbol{-(x-3)} & (\textbf{when } \boldsymbol{2 < x < 3}) \end{cases}$

(2) $\sqrt{(x-4)^2} = \begin{cases} \boldsymbol{x-4} & (\textbf{when } \boldsymbol{x \geqq 4}) \\ \boldsymbol{-(x-4)} & (\textbf{when } \boldsymbol{2 < x < 4}) \end{cases}$

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3. Rewrite the following square roots when $x > 4$.

(1) $\sqrt{(x-2)^2} = \boldsymbol{x-2}$

(2) $\sqrt{(x-4)^2} = \boldsymbol{x-4}$

(3) $\sqrt{(x-6)^2} = \begin{cases} \boldsymbol{x-6} & \textbf{(when } \boldsymbol{x \geqslant 6}\textbf{)} \\ \boldsymbol{-(x-6)} & \textbf{(when } \boldsymbol{4 < x < 6}\textbf{)} \end{cases}$

4. Rewrite the following square roots when $x < -3$.

(1) $\sqrt{(x-2)^2} = \boldsymbol{-(x-2)}$

(2) $\sqrt{(x+2)^2} = \boldsymbol{-(x+2)}$

(3) $\sqrt{(x+5)^2} = \begin{cases} \boldsymbol{x+5} & \textbf{(when } \boldsymbol{-5 \leqslant x < -3}\textbf{)} \\ \boldsymbol{-(x+5)} & \textbf{(when } \boldsymbol{x < -5}\textbf{)} \end{cases}$

J90a KUMON

Irrational Numbers II

1. If $x=\dfrac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$ and $y=\dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$, evaluate the following expressions.

(1) $$\begin{aligned} x+y &= \left(\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}\right)+\left(\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}\right) \\ &= \frac{(\sqrt{7}+\sqrt{3})^2}{4}+\frac{(\sqrt{7}-\sqrt{3})^2}{4} \\ &= \frac{10+2\sqrt{21}+10-2\sqrt{21}}{4} \\ &= \mathbf{5} \end{aligned}$$

(2) $$xy=\left(\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}\right)\left(\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}\right)=\mathbf{1}$$

(3) $$\begin{aligned} \frac{y}{x}+\frac{x}{y} &= \frac{y^2+x^2}{xy} \\ &= \frac{(x+y)^2-2xy}{xy} \\ &= 25-2=\mathbf{23} \end{aligned}$$

(4) $$\begin{aligned} \frac{x^2}{y}+\frac{y^2}{x} &= \frac{x^3+y^3}{xy} \\ &= \frac{(x+y)^3-3xy(x+y)}{xy} \\ &= 5^3-3\times1\times5 \\ &= \mathbf{110} \end{aligned}$$

J90b

2. Rewrite the following square roots.

(1) $\sqrt{(x-2)^2} = \begin{cases} \boldsymbol{x-2} & \textbf{(when } \boldsymbol{x \geqq 2}\textbf{)} \\ \boldsymbol{-(x-2)} & \textbf{(when } \boldsymbol{x < 2}\textbf{)} \end{cases}$

(2) $\sqrt{(x+5)^2} = \begin{cases} \boldsymbol{x+5} & \textbf{(when } \boldsymbol{x \geqq -5}\textbf{)} \\ \boldsymbol{-(x+5)} & \textbf{(when } \boldsymbol{x < -5}\textbf{)} \end{cases}$

(3) $\sqrt{(x-a)^2} = \begin{cases} \boldsymbol{x-a} & \textbf{(when } \boldsymbol{x \geqq a}\textbf{)} \\ \boldsymbol{-(x-a)} & \textbf{(when } \boldsymbol{x < a}\textbf{)} \end{cases}$

(4) $\sqrt{(x+2a)^2} = \begin{cases} \boldsymbol{x+2a} & \textbf{(when } \boldsymbol{x \geqq -2a}\textbf{)} \\ \boldsymbol{-(x+2a)} & \textbf{(when } \boldsymbol{x < -2a}\textbf{)} \end{cases}$

Consider this!

Rewrite the following square roots.

(1) $\sqrt{a^4}$

[Sol] $\sqrt{a^4} = \sqrt{(a^2)^2} = \boxed{\boldsymbol{a^2}}$

Whether $a \geqq 0$ or $a < 0$, then $a^2 \geqq 0$ in either case.

(2) $\sqrt{a^6}$

[Sol] $\sqrt{a^6} = \sqrt{(a^3)^2}$

The sign of a^3 depends on the sign of a.

$$\sqrt{a^6} = \begin{cases} \boxed{\boldsymbol{a^3}} & (\text{when } a \geqq 0) \\ \boxed{\boldsymbol{-a^3}} & (\text{when } a < 0) \end{cases}$$

Answers: a^2, a^3, $-a^3$

J91a KUMON

Quadratic Equations I

Solve the following quadratic equations.

Ex. 1

$x^2+3x-10=0$

[Sol] $(x+5)(x-2)=0$

$x=-5,\ 2$

Ex. 2

$6x^2-x=12$

[Sol] $6x^2-x-12=0$

$(3x+4)(2x-3)=0$

$x=-\dfrac{4}{3},\ \dfrac{3}{2}$

(1) $x^2+5x+6=0$

$(x+3)(x+2)=0$

$\boldsymbol{x=-3,\ -2}$

(2) $x^2-12x+35=0$

$(x-7)(x-5)=0$

$\boldsymbol{x=7,\ 5}$

(3) $x^2-6x=16$

$x^2-6x-16=0$

$(x-8)(x+2)=0$

$\boldsymbol{x=8,\ -2}$

(4) $x^2-2=x$

$x^2-x-2=0$

$(x-2)(x+1)=0$

$\boldsymbol{x=2,\ -1}$

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(5) $(x-2)(x-1)=0$

$\boldsymbol{x=2,\ 1}$

(9) $x^2-2x-35=0$

$(x-7)(x+5)=0$

$\boldsymbol{x=7,\ -5}$

(6) $(x-3)(x+2)=0$

$\boldsymbol{x=3,\ -2}$

(10) $x^2+2x-99=0$

$(x+11)(x-9)=0$

$\boldsymbol{x=-11,\ 9}$

(7) $(3x+2)(5x-6)=0$

$\boldsymbol{x=-\frac{2}{3},\ \frac{6}{5}}$

(11) $2x^2-17x+30=0$

$(2x-5)(x-6)=0$

$\boldsymbol{x=\frac{5}{2},\ 6}$

(8) $x(x-6)=40$

$x^2-6x-40=0$

$(x-10)(x+4)=0$

$\boldsymbol{x=10,\ -4}$

(12) $3x^2=8-10x$

$3x^2+10x-8=0$

$(3x-2)(x+4)=0$

$\boldsymbol{x=\frac{2}{3},\ -4}$

J92a KUMON

Quadratic Equations I

Solve the following quadratic equations.

(1) $x^2-8x+15=0$

$(x-5)(x-3)=0$

$\boldsymbol{x=5,\ 3}$

(2) $m^2-3m-10=0$

$(m-5)(m+2)=0$

$\boldsymbol{m=5,\ -2}$

(3) $2a^2-13a+6=0$

$(2a-1)(a-6)=0$

$\boldsymbol{a=\frac{1}{2},\ 6}$

(4) $x^2-3x=0$

$x(x-3)=0$

$\boldsymbol{x=0,\ 3}$

(5) $3x^2-x=0$

$x(3x-1)=0$

$\boldsymbol{x=0,\ \frac{1}{3}}$

(6) $2x^2=7x$

$2x^2-7x=0$

$x(2x-7)=0$

$\boldsymbol{x=0,\ \frac{7}{2}}$

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(7) $2x^2-5x+2=0$

$(2x-1)(x-2)=0$

$\boldsymbol{x=\frac{1}{2},\ 2}$

(8) $x^2+x-12=0$

$(x+4)(x-3)=0$

$\boldsymbol{x=-4,\ 3}$

(9) $x^2+x-420=0$

$(x+21)(x-20)=0$

$\boldsymbol{x=-21,\ 20}$

(10) $4x^2-x-18=0$

$(4x-9)(x+2)=0$

$\boldsymbol{x=\frac{9}{4},\ -2}$

(11) $x^2-3ax-28a^2=0$

$(x-7a)(x+4a)=0$

$\boldsymbol{x=7a,\ -4a}$

(12) $2x^2-ax-a^2=0$

$(2x+a)(x-a)=0$

$\boldsymbol{x=-\frac{a}{2},\ a}$

J93a KUMON

Quadratic Equations I

Solve the following quadratic equations.

Ex.

① $4x^2-25=0$

[Sol 1] $(2x+5)(2x-5)=0$

$x=-\frac{5}{2}, \frac{5}{2}$

[Sol 2] $4x^2=25$

$x^2=\frac{25}{4}$

$x=\pm\frac{5}{2}$

② $x^2=18$

[Sol] $x=\pm\sqrt{18}$

$=\pm 3\sqrt{2}$

③ $2x^2=3$

[Sol] $x^2=\frac{3}{2}$

$x=\pm\sqrt{\frac{3}{2}}=\pm\frac{\sqrt{6}}{2}$

(1) $4x^2-9=0$

$(2x+3)(2x-3)=0$

$\boldsymbol{x=-\frac{3}{2}, \frac{3}{2}}$

$\left[\boldsymbol{x=\pm\frac{3}{2}}\right]$

(2) $4x^2=81$

$x^2=\frac{81}{4}$

$\boldsymbol{x=\pm\frac{9}{2}}$

(3) $9x^2=144$

$x^2=16$

$\boldsymbol{x=\pm 4}$

(4) $x^2=12$

$\boldsymbol{x}=\pm\sqrt{12}$

$\boldsymbol{=\pm 2\sqrt{3}}$

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(5) $x^2=50$

$$\boldsymbol{x}=\pm\sqrt{50}$$
$$\mathbf{=\pm 5\sqrt{2}}$$

(6) $2x^2-5=0$

$$2x^2=5$$
$$x^2=\frac{5}{2}$$
$$\boldsymbol{x}=\pm\sqrt{\frac{5}{2}}$$
$$\mathbf{=\pm\frac{\sqrt{10}}{2}}$$

(7) $(x+8)(x-8)=225$

$$x^2-64=225$$
$$x^2=289$$
$$\boldsymbol{x}\mathbf{=\pm 17}$$

(8) $2(x+5)(x-5)+21=7$

$$2(x^2-25)=-14$$
$$x^2-25=-7$$
$$x^2=18$$
$$\boldsymbol{x}=\pm\sqrt{18}$$
$$\mathbf{=\pm 3\sqrt{2}}$$

(9) $(2+3x)^2+(2-3x)^2=26$

$$4+12x+9x^2+4-12x+9x^2=26$$
$$18x^2+8=26$$
$$18x^2=18$$
$$x^2=1$$
$$\boldsymbol{x}\mathbf{=\pm 1}$$

(10) $\left(\frac{x+1}{2}\right)^2-\frac{x}{2}=2$

$$\frac{x^2+2x+1}{4}-\frac{x}{2}=2$$
$$x^2+2x+1-2x=8$$
$$x^2=7$$
$$\boldsymbol{x}\mathbf{=\pm\sqrt{7}}$$

J94a KUMON

Quadratic Equations I

Solve the following quadratic equations.

Ex. 1

$(x-2)^2=5$

[Sol] $x-2=\pm\sqrt{5}$

$x=2\pm\sqrt{5}$

Ex. 2

$(x-3)^2=25$

[Sol] $x-3=\pm 5$

$x=3\pm 5$

$x=8,\ -2$

(1) $(x-1)^2=7$

$x-1=\pm\sqrt{7}$

$\boldsymbol{x=1\pm\sqrt{7}}$

(2) $(x+3)^2=5$

$x+3=\pm\sqrt{5}$

$\boldsymbol{x=-3\pm\sqrt{5}}$

(3) $x^2-6x+9=2$

$(x-3)^2=2$

$x-3=\pm\sqrt{2}$

$\boldsymbol{x=3\pm\sqrt{2}}$

(4) $(x+3)^2=4$

$x+3=\pm 2$

$x=-3\pm 2$

$\boldsymbol{x=-1,\ -5}$

(5) $(x-5)^2=25$

$x-5=\pm 5$

$x=5\pm 5$

$\boldsymbol{x=10,\ 0}$

(6) $(x-3)^2-9=0$

$x-3=\pm 3$

$x=3\pm 3$

$\boldsymbol{x=6,\ 0}$

J94b

(7) $(2x-3)^2=25$

$$2x-3=\pm 5$$
$$2x=3\pm 5$$
$$x=\frac{3\pm 5}{2}$$
$$\boldsymbol{x=4,\ -1}$$

(8) $(3x+5)^2=4$

$$3x+5=\pm 2$$
$$3x=-5\pm 2$$
$$x=\frac{-5\pm 2}{3}$$
$$\boldsymbol{x=-1,\ -\frac{7}{3}}$$

(9) $(2x-3)^2-4=0$

$$(2x-3)^2=4$$
$$2x-3=\pm 2$$
$$2x=3\pm 2$$
$$x=\frac{3\pm 2}{2}$$
$$\boldsymbol{x=\frac{5}{2},\ \frac{1}{2}}$$

(10)* $(2x-3)^2=a \quad (a\geqq 0)$

$$2x-3=\pm\sqrt{a}$$
$$2x=3\pm\sqrt{a}$$
$$\boldsymbol{x=\frac{3\pm\sqrt{a}}{2}}$$

(11) $(ax+5)^2=b \quad (a\neq 0,\ b\geqq 0)$

$$ax+5=\pm\sqrt{b}$$
$$ax=-5\pm\sqrt{b}$$
$$\boldsymbol{x=\frac{-5\pm\sqrt{b}}{a}}$$

(12) $(ax-b)^2=c^2 \quad (a\neq 0)$

$$ax-b=\pm c$$
$$ax=b\pm c$$
$$\boldsymbol{x=\frac{b\pm c}{a}}$$

J95a KUMON

Quadratic Equations I

Solve the following equations by completing the square.

Ex.

$x^2-6x-1=0$

[Sol] $x^2-6x=1$

$x^2-6x+9=1+9$

Add $3^2=9$ to both sides. This is called ***completing the square***.

$(x-3)^2=10$

$x-3=\pm\sqrt{10}$

$x=3\pm\sqrt{10}$

(1) $x^2-6x-3=0$

$$x^2-6x=3$$
$$x^2-6x+9=3+9$$
$$(x-3)^2=12$$
$$x-3=\pm\sqrt{12}$$
$$=\pm 2\sqrt{3}$$
$$\boldsymbol{x=3\pm 2\sqrt{3}}$$

It may help to write this step as $x^2-2\cdot 3\cdot x+3^2=3+3^2$.

(2) $x^2-8x+5=0$

$$x^2-8x=-5$$
$$x^2-8x+16=-5+16$$
$$(x-4)^2=11$$
$$x-4=\pm\sqrt{11}$$
$$\boldsymbol{x=4\pm\sqrt{11}}$$

(3) $x^2+6x+6=0$

$$x^2+6x=-6$$
$$x^2+6x+9=-6+9$$
$$(x+3)^2=3$$
$$x+3=\pm\sqrt{3}$$
$$\boldsymbol{x=-3\pm\sqrt{3}}$$

(4) $x^2-2ax+1=0$

[Sol] $x^2-2ax=-1$

$$x^2-2ax+\boxed{\boldsymbol{a^2}}=-1+\boxed{\boldsymbol{a^2}}$$
$$(\ \boldsymbol{x-a}\ )^2=a^2-1$$
$$x-a=\pm\sqrt{\boxed{\boldsymbol{a^2-1}}}$$
$$x=\boldsymbol{a\pm\sqrt{a^2-1}}$$

J95b

Ex.

$x^2 - 5x - 7 = 0$

[Sol] $x^2 - 5x = 7$

Add $\left(\frac{5}{2}\right)^2$ to both sides.

$$x^2 - 5x + \left(\frac{5}{2}\right)^2 = 7 + \left(\frac{5}{2}\right)^2$$

$$\left(x - \frac{5}{2}\right)^2 = \frac{53}{4}$$

$$x - \frac{5}{2} = \pm\frac{\sqrt{53}}{2}$$

$$x = \frac{5 \pm \sqrt{53}}{2}$$

(5) $x^2 - 3x - 7 = 0$

$$x^2 - 3x = 7$$

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = 7 + \left(\frac{3}{2}\right)^2$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{37}{4}$$

$$x - \frac{3}{2} = \pm\frac{\sqrt{37}}{2}$$

$$\boldsymbol{x = \frac{3 \pm \sqrt{37}}{2}}$$

(6) $x^2 + x - 1 = 0$

$$x^2 + x = 1$$

$$x^2 + x + \left(\frac{1}{2}\right)^2 = 1 + \left(\frac{1}{2}\right)^2$$

$$\left(x + \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$x + \frac{1}{2} = \pm\frac{\sqrt{5}}{2}$$

$$\boldsymbol{x = \frac{-1 \pm \sqrt{5}}{2}}$$

(7) $x^2 + \frac{1}{4}x - \frac{1}{2} = 0$

$$x^2 + \frac{1}{4}x = \frac{1}{2}$$

$$x^2 + \frac{1}{4}x + \left(\frac{1}{8}\right)^2 = \frac{1}{2} + \left(\frac{1}{8}\right)^2$$

$$\left(x + \frac{1}{8}\right)^2 = \frac{33}{64}$$

$$x + \frac{1}{8} = \pm\frac{\sqrt{33}}{8}$$

$$\boldsymbol{x = \frac{-1 \pm \sqrt{33}}{8}}$$

(8) $x^2 + ax - 4 = 0$

[Sol] $x^2 + ax = 4$

$$x^2 + ax + \left(\boldsymbol{\frac{a}{2}}\right)^2 = 4 + \left(\boldsymbol{\frac{a}{2}}\right)^2$$

$$\left(\boldsymbol{x + \frac{a}{2}}\right)^2 = \frac{16 + \boxed{\boldsymbol{a^2}}}{4}$$

$$x + \frac{a}{2} = \pm\frac{\sqrt{16 + a^2}}{2}$$

$$x = \boldsymbol{\frac{-a \pm \sqrt{16 + a^2}}{2}}$$

J96a KUMON

Quadratic Equations I

Solve the following equations by completing the square.

Ex.

$2x^2-3x-1=0$

[Sol] $x^2-\frac{3}{2}x=\frac{1}{2}$

$$x^2-\frac{3}{2}x+\left(\frac{3}{4}\right)^2=\frac{1}{2}+\left(\frac{3}{4}\right)^2$$

$$\left(x-\frac{3}{4}\right)^2=\frac{17}{16}$$

$$x-\frac{3}{4}=\pm\frac{\sqrt{17}}{4}$$

$$x=\frac{3\pm\sqrt{17}}{4}$$

(1) $3x^2+5x-3=0$

$$x^2+\frac{5}{3}x=1$$

$$x^2+\frac{5}{3}x+\left(\frac{5}{6}\right)^2=1+\left(\frac{5}{6}\right)^2$$

$$\left(x+\frac{5}{6}\right)^2=\frac{61}{36}$$

$$x+\frac{5}{6}=\pm\frac{\sqrt{61}}{6}$$

$$\boldsymbol{x=\frac{-5\pm\sqrt{61}}{6}}$$

(2) $4x^2+3x-2=0$

$$x^2+\frac{3}{4}x=\frac{1}{2}$$

$$x^2+\frac{3}{4}x+\left(\frac{3}{8}\right)^2=\frac{1}{2}+\left(\frac{3}{8}\right)^2$$

$$\left(x+\frac{3}{8}\right)^2=\frac{41}{64}$$

$$x+\frac{3}{8}=\pm\frac{\sqrt{41}}{8}$$

$$\boldsymbol{x=\frac{-3\pm\sqrt{41}}{8}}$$

(3) $4x^2-5x-3=0$

$$x^2-\frac{5}{4}x=\frac{3}{4}$$

$$x^2-\frac{5}{4}x+\left(\frac{5}{8}\right)^2=\frac{3}{4}+\left(\frac{5}{8}\right)^2$$

$$\left(x-\frac{5}{8}\right)^2=\frac{73}{64}$$

$$x-\frac{5}{8}=\pm\frac{\sqrt{73}}{8}$$

$$\boldsymbol{x=\frac{5\pm\sqrt{73}}{8}}$$

J96b

Ex.

$3x^2-5x+2=0$

[Sol] $x^2-\frac{5}{3}x=-\frac{2}{3}$

$$x^2-\frac{5}{3}x+\left(\frac{5}{6}\right)^2=-\frac{2}{3}+\left(\frac{5}{6}\right)^2$$

$$\left(x-\frac{5}{6}\right)^2=\frac{1}{36}$$

$$x-\frac{5}{6}=\pm\frac{1}{6}$$

$$x=\frac{5}{6}\pm\frac{1}{6}$$

$$x=1,\ \frac{2}{3}$$

(4) $3x^2-2x-1=0$

$$x^2-\frac{2}{3}x=\frac{1}{3}$$

$$x^2-\frac{2}{3}x+\left(\frac{1}{3}\right)^2=\frac{1}{3}+\left(\frac{1}{3}\right)^2$$

$$\left(x-\frac{1}{3}\right)^2=\frac{4}{9}$$

$$x-\frac{1}{3}=\pm\frac{2}{3}$$

$$x=\frac{1}{3}\pm\frac{2}{3}$$

$$\boldsymbol{x=1,\ -\frac{1}{3}}$$

(5) $4x^2-4x-3=0$

$$x^2-x=\frac{3}{4}$$

$$x^2-x+\left(\frac{1}{2}\right)^2=\frac{3}{4}+\left(\frac{1}{2}\right)^2$$

$$\left(x-\frac{1}{2}\right)^2=1$$

$$x-\frac{1}{2}=\pm1$$

$$x=\frac{1}{2}\pm1$$

$$\boldsymbol{x=\frac{3}{2},\ -\frac{1}{2}}$$

(6) $x^2-2ax=b$

$$x^2-2ax+a^2=b+a^2$$

$$(x-a)^2=b+a^2$$

$$x-a=\pm\sqrt{b+a^2}$$

$$\boldsymbol{x=a\pm\sqrt{b+a^2}}$$

J 97a KUMON

Quadratic Equations I

1. Solve the following equations by completing the square.

(1) $3x^2-7x+2=0$

$$x^2-\frac{7}{3}x=-\frac{2}{3}$$

$$x^2-\frac{7}{3}x+\left(\frac{7}{6}\right)^2=-\frac{2}{3}+\left(\frac{7}{6}\right)^2$$

$$\left(x-\frac{7}{6}\right)^2=\frac{25}{36}$$

$$x-\frac{7}{6}=\pm\frac{5}{6}$$

$$x=\frac{7}{6}\pm\frac{5}{6}$$

$$\boldsymbol{x=2,\ \frac{1}{3}}$$

(3)* $ax^2+bx+c=0\quad(a\neq 0)$

[Sol] $x^2+\frac{b}{a}x=-\frac{c}{a}$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

$$\left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

(2) $3x^2-7x+1=0$

$$x^2-\frac{7}{3}x=-\frac{1}{3}$$

$$x^2-\frac{7}{3}x+\left(\frac{7}{6}\right)^2=-\frac{1}{3}+\left(\frac{7}{6}\right)^2$$

$$\left(x-\frac{7}{6}\right)^2=\frac{37}{36}$$

$$x-\frac{7}{6}=\pm\frac{\sqrt{37}}{6}$$

$$\boldsymbol{x=\frac{7\pm\sqrt{37}}{6}}$$

J97b

From J97a, question 1.(3):

Quadratic Formula

When $ax^2+bx+c=0$, $\quad x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

2. Solve the following equations using the quadratic formula.

Ex.

$3x^2+9x+2=0$

[Sol] $a=3, b=9, c=2$

$$x=\frac{-9\pm\sqrt{9^2-4\times3\times2}}{2\times3}$$

$$=\frac{-9\pm\sqrt{81-24}}{6}$$

$$=\frac{-9\pm\sqrt{57}}{6}$$

(1) $3x^2-9x+2=0$

[Sol] $a=3,\ b=\boxed{\mathbf{-9}},\ c=2$

$$x=\frac{9\pm\sqrt{(-9)^2-4\times3\times2}}{2\times3}$$

$$=\frac{9\pm\sqrt{81-24}}{6}$$

$$=\boldsymbol{\frac{9\pm\sqrt{57}}{6}}$$

(2) $2x^2+7x+1=0$

$a=2,\ b=7,\ c=1$

$$\boldsymbol{x}=\frac{-7\pm\sqrt{7^2-4\times2\times1}}{2\times2}$$

$$=\frac{-7\pm\sqrt{49-8}}{4}$$

$$=\boldsymbol{\frac{-7\pm\sqrt{41}}{4}}$$

(3) $2x^2-7x-1=0$

$a=2,\ b=-7,\ c=-1$

$$\boldsymbol{x}=\frac{7\pm\sqrt{(-7)^2-4\times2\times(-1)}}{2\times2}$$

$$=\frac{7\pm\sqrt{49+8}}{4}$$

$$=\boldsymbol{\frac{7\pm\sqrt{57}}{4}}$$

J98a KUMON

Quadratic Equations I

Quadratic Formula

When $ax^2+bx+c=0$, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Solve the following equations using the quadratic formula.

Ex.

$3x^2+7x+2=0$

[Sol] $x=\dfrac{-7\pm\sqrt{7^2-4\times3\times2}}{2\times3}=\dfrac{-7\pm\sqrt{25}}{6}=\dfrac{-7\pm5}{6}$

$x=-\dfrac{1}{3}, -2$

(1) $3x^2-7x+2=0$

$$x=\frac{7\pm\sqrt{(-7)^2-4\times3\times2}}{2\times3}=\frac{7\pm\sqrt{25}}{6}=\frac{7\pm5}{6}$$

$\boldsymbol{x=2,\ \dfrac{1}{3}}$

(2) $3x^2+5x-8=0$

$$x=\frac{-5\pm\sqrt{5^2-4\times3\times(-8)}}{2\times3}=\frac{-5\pm\sqrt{121}}{6}=\frac{-5\pm11}{6}$$

$\boldsymbol{x=1,\ -\dfrac{8}{3}}$

(3) $3x^2-5x-8=0$

$$x=\frac{5\pm\sqrt{(-5)^2-4\times3\times(-8)}}{2\times3}=\frac{5\pm\sqrt{121}}{6}=\frac{5\pm11}{6}$$

$\boldsymbol{x=\dfrac{8}{3},\ -1}$

(4) $2x^2+7x+3=0$

$$x=\frac{-7\pm\sqrt{7^2-4\times2\times3}}{2\times2}=\frac{-7\pm\sqrt{25}}{4}=\frac{-7\pm5}{4}$$

$\boldsymbol{x=-\dfrac{1}{2},\ -3}$

J98b

(5) $2x^2+5x+3=0$

$$x=\frac{-5\pm\sqrt{5^2-4\times2\times3}}{2\times2}$$
$$=\frac{-5\pm1}{4}$$
$$\boldsymbol{x=-1,\ -\frac{3}{2}}$$

(8) $2x^2-7x+1=0$

$$\boldsymbol{x}=\frac{7\pm\sqrt{(-7)^2-4\times2\times1}}{2\times2}$$
$$\boldsymbol{=\frac{7\pm\sqrt{41}}{4}}$$

(6) $2x^2-5x+3=0$

$$x=\frac{5\pm\sqrt{(-5)^2-4\times2\times3}}{2\times2}$$
$$=\frac{5\pm1}{4}$$
$$\boldsymbol{x=\frac{3}{2},\ 1}$$

(9) $4x^2-9x+3=0$

$$\boldsymbol{x}=\frac{9\pm\sqrt{(-9)^2-4\times4\times3}}{2\times4}$$
$$\boldsymbol{=\frac{9\pm\sqrt{33}}{8}}$$

(7) $2x^2+7x-1=0$

$$x=\frac{-7\pm\sqrt{7^2-4\times2\times(-1)}}{2\times2}$$
$$\boldsymbol{x=\frac{-7\pm\sqrt{57}}{4}}$$

(10) $3x^2+7x-6=0$

$$x=\frac{-7\pm\sqrt{7^2-4\times3\times(-6)}}{2\times3}$$
$$=\frac{-7\pm\sqrt{121}}{6}$$
$$=\frac{-7\pm11}{6}$$
$$\boldsymbol{x=\frac{2}{3},\ -3}$$

J99a KUMON

Quadratic Equations I

Quadratic Formula

When $ax^2+bx+c=0$, $\quad x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Solve the following equations using the quadratic formula.

Ex.

$-2x^2-7x+3=0$

[Sol] $2x^2+7x-3=0$ Multiply both sides by -1.

$x=\dfrac{-7\pm\sqrt{73}}{4}$

(1) $-2x^2-7x-3=0$

$2x^2+7x+3=0$

$x=\dfrac{-7\pm\sqrt{25}}{4}$

$=\dfrac{-7\pm5}{4}$

$\boldsymbol{x=-\dfrac{1}{2},\ -3}$

(2) $-2x^2+5x-2=0$

$2x^2-5x+2=0$

$x=\dfrac{5\pm\sqrt{9}}{4}$

$=\dfrac{5\pm3}{4}$

$\boldsymbol{x=2,\ \dfrac{1}{2}}$

(3) $-4x^2+9x+2=0$

$4x^2-9x-2=0$

$\boldsymbol{x=\dfrac{9\pm\sqrt{113}}{8}}$

(4) $-2x^2-7x-1=0$

$2x^2+7x+1=0$

$\boldsymbol{x=\dfrac{-7\pm\sqrt{41}}{4}}$

J99b

(5) $3x^2+5x+1=0$

$$\boldsymbol{x=\frac{-5\pm\sqrt{13}}{6}}$$

(6) $3x^2+5x+2=0$

$$x=\frac{-5\pm1}{6}$$

$$\boldsymbol{x=-\frac{2}{3},\ -1}$$

(7) $2x^2+5x-1=0$

$$\boldsymbol{x=\frac{-5\pm\sqrt{33}}{4}}$$

(8) $-12x^2+11x-2=0$

$$12x^2-11x+2=0$$

$$x=\frac{11\pm\sqrt{25}}{24}$$

$$=\frac{11\pm5}{24}$$

$$\boldsymbol{x=\frac{2}{3},\ \frac{1}{4}}$$

(9) $-3x^2+7x-2=0$

$$3x^2-7x+2=0$$

$$x=\frac{7\pm\sqrt{25}}{6}$$

$$=\frac{7\pm5}{6}$$

$$\boldsymbol{x=2,\ \frac{1}{3}}$$

(10) $-3x^2-7x+5=0$

$$3x^2+7x-5=0$$

$$\boldsymbol{x=\frac{-7\pm\sqrt{109}}{6}}$$

J 100a KUMON

Quadratic Equations I

Solve the following equations using the quadratic formula.

(1) $3x^2+9x+5=0$

$$\boldsymbol{x=\frac{-9\pm\sqrt{21}}{6}}$$

(4) $3x^2-9x+5=0$

$$\boldsymbol{x=\frac{9\pm\sqrt{21}}{6}}$$

(2) $2x^2+7x+3=0$

$$x=\frac{-7\pm\sqrt{25}}{4}$$

$$=\frac{-7\pm5}{4}$$

$$\boldsymbol{x=-\frac{1}{2},\ -3}$$

(5) $8x=3x^2+5$

$$3x^2-8x+5=0$$

$$x=\frac{8\pm\sqrt{4}}{6}$$

$$=\frac{8\pm2}{6}$$

$$\boldsymbol{x=\frac{5}{3},\ 1}$$

(3) $-x^2-7x+3=0$

$$x^2+7x-3=0$$

$$\boldsymbol{x=\frac{-7\pm\sqrt{61}}{2}}$$

J100b

(6) $9x^2-13x+4=0$

$$x=\frac{13\pm\sqrt{25}}{18}$$

$$=\frac{13\pm5}{18}$$

$$\boldsymbol{x=1,\ \frac{4}{9}}$$

(9) $x^2-7x+1=0$

$$x=\frac{7\pm\sqrt{45}}{2}$$

$$\boldsymbol{=\frac{7\pm3\sqrt{5}}{2}}$$

(7) $-x^2-5x+3=0$

$$x^2+5x-3=0$$

$$\boldsymbol{x=\frac{-5\pm\sqrt{37}}{2}}$$

(10) $3x^2-11x+7=0$

$$\boldsymbol{x=\frac{11\pm\sqrt{37}}{6}}$$

(8) $-3x^2-8x-5=0$

$$3x^2+8x+5=0$$

$$x=\frac{-8\pm\sqrt{4}}{6}$$

$$=\frac{-8\pm2}{6}$$

$$\boldsymbol{x=-1,\ -\frac{5}{3}}$$

J 101a KUMON

Quadratic Equations II

Formula

When $ax^2+bx+c=0$, $\quad x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Solve the following equations using the quadratic formula.

(1) $3x^2+9x+2=0$

$$\boldsymbol{x}=\frac{-9\pm\sqrt{9^2-4\times3\times2}}{2\times3}$$

$$=\boldsymbol{\frac{-9\pm\sqrt{57}}{6}}$$

(2) $2x^2+7x+1=0$

$$\boldsymbol{x}=\frac{-7\pm\sqrt{7^2-4\times2\times1}}{2\times2}$$

$$=\boldsymbol{\frac{-7\pm\sqrt{41}}{4}}$$

(3) $4x^2+9x+3=0$

$$\boldsymbol{x}=\frac{-9\pm\sqrt{9^2-4\times4\times3}}{2\times4}$$

$$=\boldsymbol{\frac{-9\pm\sqrt{33}}{8}}$$

(4) $3x^2-9x+2=0$

$$\boldsymbol{x}=\frac{9\pm\sqrt{(-9)^2-4\times3\times2}}{2\times3}$$

$$=\boldsymbol{\frac{9\pm\sqrt{57}}{6}}$$

(5) $2x^2+7x-1=0$

$$\boldsymbol{x}=\frac{-7\pm\sqrt{7^2-4\times2\times(-1)}}{2\times2}$$

$$=\boldsymbol{\frac{-7\pm\sqrt{57}}{4}}$$

(6) $3x^2+7x+2=0$

$$x=\frac{-7\pm\sqrt{7^2-4\times3\times2}}{2\times3}$$

$$=\frac{-7\pm\sqrt{25}}{6}$$

$$=\frac{-7\pm5}{6}$$

$$\boldsymbol{x=-\frac{1}{3},\ -2}$$

(7) $3x^2-7x+2=0$

$$x=\frac{7\pm\sqrt{(-7)^2-4\times3\times2}}{2\times3}$$

$$=\frac{7\pm\sqrt{25}}{6}$$

$$=\frac{7\pm5}{6}$$

$$\boldsymbol{x=2,\frac{1}{3}}$$

(8) $3x^2+5x-8=0$

$$x=\frac{-5\pm\sqrt{5^2-4\times3\times(-8)}}{2\times3}$$

$$=\frac{-5\pm\sqrt{121}}{6}$$

$$=\frac{-5\pm11}{6}$$

$$\boldsymbol{x=1,-\frac{8}{3}}$$

(9) $3x^2+7x-6=0$

$$x=\frac{-7\pm\sqrt{7^2-4\times3\times(-6)}}{2\times3}$$

$$=\frac{-7\pm\sqrt{121}}{6}$$

$$=\frac{-7\pm11}{6}$$

$$\boldsymbol{x=\frac{2}{3},\ -3}$$

(10)* $ax^2+2b'x+c=0 \quad (a\neq0)$

(Substitute a, $2b'$, c into the quadratic formula.)

[Sol] $x=\dfrac{-2b'\pm\sqrt{(\ \boldsymbol{2b'}\ )^2-4ac}}{2a}$

$$=\frac{-2b'\pm2\sqrt{\boxed{\boldsymbol{b'^2-ac}}}}{2a}$$

$$=\boldsymbol{\frac{-b'\pm\sqrt{b'^2-ac}}{a}}$$

J 102a KUMON

Quadratic Equations II

From the result of J 101 b question (10), we can derive the Quadratic Formula II, which can be used when the coefficient of x is an even number.

Quadratic Formula II

When $ax^2+2b'x+c=0$, $\quad x=\dfrac{-b'\pm\sqrt{b'^2-ac}}{a}$

Ex.

$3x^2+10x+1=0$

[Sol] $x=\dfrac{-5\pm\sqrt{5^2-3\times1}}{3}$ ☜ From $b'=5$.

$=\dfrac{-5\pm\sqrt{22}}{3}$

1. Solve the following equations using the quadratic formula II.

(1) $3x^2+14x+5=0$

$x=\dfrac{-7\pm\sqrt{7^2-3\times5}}{3}$

$=\dfrac{-7\pm\sqrt{34}}{3}$

(2) $3x^2+8x+5=0$

$x=\dfrac{-4\pm\sqrt{4^2-3\times5}}{3}$

$=\dfrac{-4\pm1}{3}$

$x=-1,\ -\dfrac{5}{3}$

(3) $x^2-8x+5=0$

$x=\dfrac{4\pm\sqrt{(-4)^2-1\times5}}{1}$

$=4\pm\sqrt{11}$

(4) $x^2+6x-3=0$

$x=\dfrac{-3\pm\sqrt{3^2-1\times(-3)}}{1}$

$=-3\pm2\sqrt{3}$

J102b

Quadratic Formula I

When $ax^2+bx+c=0$,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Quadratic Formula II

When $ax^2+2b'x+c=0$,

$$x=\frac{-b'\pm\sqrt{b'^2-ac}}{a}$$

2. Solve the following equations using either quadratic formula.

(1) $4x^2+9x+1=0$

$$\boldsymbol{x}=\frac{-9\pm\sqrt{9^2-4\times4\times1}}{2\times4}$$

$$\boldsymbol{=\frac{-9\pm\sqrt{65}}{8}}$$

(2) $3x^2-10x+3=0$

$$x=\frac{5\pm\sqrt{(-5)^2-3\times3}}{3}$$

$$=\frac{5\pm4}{3}$$

$$\boldsymbol{x=3,\ \frac{1}{3}}$$

(3) $x^2+8x-3=0$

$$\boldsymbol{x}=\frac{-4\pm\sqrt{4^2-1\times(-3)}}{1}$$

$$\boldsymbol{=-4\pm\sqrt{19}}$$

(4) $-2x^2-7x+3=0$

$$2x^2+7x-3=0$$

$$\boldsymbol{x}=\frac{-7\pm\sqrt{7^2-4\times2\times(-3)}}{2\times2}$$

$$\boldsymbol{=\frac{-7\pm\sqrt{73}}{4}}$$

(5) $-2x^2-6x+7=0$

$$2x^2+6x-7=0$$

$$\boldsymbol{x}=\frac{-3\pm\sqrt{3^2-2\times(-7)}}{2}$$

$$\boldsymbol{=\frac{-3\pm\sqrt{23}}{2}}$$

(6) $-2x^2-7x-3=0$

$$2x^2+7x+3=0$$

$$x=\frac{-7\pm\sqrt{7^2-4\times2\times3}}{2\times2}$$

$$=\frac{-7\pm5}{4}$$

$$\boldsymbol{x=-\frac{1}{2},\ -3}$$

J 103a KUMON

Quadratic Equations II

Quadratic Formula I

When $ax^2+bx+c=0$,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Quadratic Formula II

When $ax^2+2b'x+c=0$,

$$x=\frac{-b'\pm\sqrt{b'^2-ac}}{a}$$

Solve the following equations using either quadratic formula.

(1) $3x^2+14x+5=0$

$$\boldsymbol{x}=\frac{-7\pm\sqrt{7^2-3\times5}}{3}$$

$$=\frac{\mathbf{-7\pm\sqrt{34}}}{\mathbf{3}}$$

(2) $2x^2+6x+3=0$

$$\boldsymbol{x}=\frac{-3\pm\sqrt{3^2-2\times3}}{2}$$

$$=\frac{\mathbf{-3\pm\sqrt{3}}}{\mathbf{2}}$$

(3) $-x^2-8x+3=0$

$$x^2+8x-3=0$$

$$\boldsymbol{x}=\frac{-4\pm\sqrt{4^2-1\times(-3)}}{1}$$

$$=\mathbf{-4\pm\sqrt{19}}$$

(4) $3x=3x^2-6$

Take out the common factor 3.

$$3x^2-3x-6=0$$

$$x^2-x-2=0$$

$$x=\frac{1\pm\sqrt{(-1)^2-4\times1\times(-2)}}{2\times1}$$

$$=\frac{1\pm3}{2}$$

$$\boldsymbol{x}=\mathbf{2,\ -1}$$

J103b

(5) $4x+5=3x^2$

$3x^2-4x-5=0$

$\boldsymbol{x}=\dfrac{2\pm\sqrt{(-2)^2-3\times(-5)}}{3}$

$\boldsymbol{=\dfrac{2\pm\sqrt{19}}{3}}$

(8) $x^2-10x+1=0$

$\boldsymbol{x}=\dfrac{5\pm\sqrt{(-5)^2-1\times1}}{1}$

$=5\pm\sqrt{24}$

$\boldsymbol{=5\pm2\sqrt{6}}$

(6) $3x^2-8x-4=0$

$\boldsymbol{x}=\dfrac{4\pm\sqrt{(-4)^2-3\times(-4)}}{3}$

$=\dfrac{4\pm\sqrt{28}}{3}$

$\boldsymbol{=\dfrac{4\pm2\sqrt{7}}{3}}$

(9) $-3x^2-8x-5=0$

$3x^2+8x+5=0$

$x=\dfrac{-4\pm\sqrt{4^2-3\times5}}{3}$

$=\dfrac{-4\pm1}{3}$

$\boldsymbol{x=-1,\ -\dfrac{5}{3}}$

(7) $9x^2-12x+4=0$

$\boldsymbol{x}=\dfrac{6\pm\sqrt{(-6)^2-9\times4}}{9}$

$=\dfrac{6\pm0}{9}$

$\boldsymbol{=\dfrac{2}{3}}$

(10) $3x^2-9x+4=0$

$\boldsymbol{x}=\dfrac{9\pm\sqrt{(-9)^2-4\times3\times4}}{2\times3}$

$\boldsymbol{=\dfrac{9\pm\sqrt{33}}{6}}$

J 104a KUMON

Quadratic Equations II

Solve the following equations either by factorisation or using the quadratic formula.

(1) $x^2-8x+12=0$

$(x-6)(x-2)=0$

$\boldsymbol{x=6,\ 2}$

(2) $x^2+9=6x$

$x^2-6x+9=0$

$(x-3)^2=0$

$\boldsymbol{x=3}$

(3) $-9x^2+6x-1=0$

$9x^2-6x+1=0$

$(3x-1)^2=0$

$\boldsymbol{x=\frac{1}{3}}$

(4) $5(x^2+2x-1)=-5$

$x^2+2x-1=-1$

$x^2+2x=0$

$x(x+2)=0$

$\boldsymbol{x=0,\ -2}$

(5) $3x^2+5x=8$

$3x^2+5x-8=0$

$(3x+8)(x-1)=0$

$\boldsymbol{x=-\frac{8}{3},\ 1}$

J104b

(6) $3x^2 = 4 - 5x$

$3x^2 + 5x - 4 = 0$

$$\boldsymbol{x} = \frac{-5 \pm \sqrt{5^2 - 4 \times 3 \times (-4)}}{2 \times 3}$$

$$= \boldsymbol{\frac{-5 \pm \sqrt{73}}{6}}$$

(8) $3x^2 - 4x - 1 = 0$

$$\boldsymbol{x} = \frac{2 \pm \sqrt{(-2)^2 - 3 \times (-1)}}{3}$$

$$= \boldsymbol{\frac{2 \pm \sqrt{7}}{3}}$$

(7) $3x^2 - 4x + 7 = 5x^2 - 7x + 8$

$2x^2 - 3x + 1 = 0$

$(2x - 1)(x - 1) = 0$

$\boldsymbol{x = \frac{1}{2},\ 1}$

(9) $(x + 2)^2 - 3(x + 2) - 4 = 0$

$x^2 + 4x + 4 - 3x - 6 - 4 = 0$

$x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

$\boldsymbol{x = -3,\ 2}$

Alternative Solution

$[(x + 2) - 4][(x + 2) + 1] = 0$

$(x - 2)(x + 3) = 0$

$\boldsymbol{x = 2,\ -3}$

J 105a KUMON

Quadratic Equations II

Solve the following equations either by factorisation or using the quadratic formula.

(1) $4x^2-15x+14=0$

$(4x-7)(x-2)=0$

$\boldsymbol{x=\frac{7}{4},\ 2}$

(2) $15x^2-23x-28=0$

$(5x+4)(3x-7)=0$

$\boldsymbol{x=-\frac{4}{5},\ \frac{7}{3}}$

(3) $3x^2+16x-35=0$

$(3x-5)(x+7)=0$

$\boldsymbol{x=\frac{5}{3},\ -7}$

(4) $-4x^2-16x+9=0$

$4x^2+16x-9=0$

$(2x+9)(2x-1)=0$

$\boldsymbol{x=-\frac{9}{2},\ \frac{1}{2}}$

(5) $x(x+1)=56$

$x^2+x=56$

$x^2+x-56=0$

$(x+8)(x-7)=0$

$\boldsymbol{x=-8,\ 7}$

J105b

Multiply both sides by 3.

(6) $x^2=\dfrac{2}{3}(x+1)$

$3x^2=2(x+1)$
$3x^2=2x+2$
$3x^2-2x-2=0$

$\boldsymbol{x}=\dfrac{1\pm\sqrt{(-1)^2-3\times(-2)}}{3}$

$\quad=\boldsymbol{\dfrac{1\pm\sqrt{7}}{3}}$

Multiply both sides by 6.

(7) $\dfrac{x(x+1)}{2}=\dfrac{2x-3}{3}+1$

$3x(x+1)=2(2x-3)+6$
$3x^2+3x=4x$
$3x^2-x=0$
$x(3x-1)=0$

$\boldsymbol{x=0,\ \dfrac{1}{3}}$

Multiply both sides by 12.

(8) $\dfrac{x^2}{4}+\dfrac{x}{3}-\dfrac{1}{6}=0$

$3x^2+4x-2=0$

$\boldsymbol{x}=\dfrac{-2\pm\sqrt{2^2-3\times(-2)}}{3}$

$\quad=\boldsymbol{\dfrac{-2\pm\sqrt{10}}{3}}$

Multiply both sides by 10.

(9) $0.2x^2+0.7x+0.3=0$

$2x^2+7x+3=0$
$(2x+1)(x+3)=0$

$\boldsymbol{x=-\dfrac{1}{2},\ -3}$

J 106a KUMON

Quadratic Equations II

Solve each equation both by using the quadratic formula and by factorisation.

(1) $6x^2+13x+6=0$

Quadratic Formula

$$x=\frac{-13\pm\sqrt{169-4\times6\times6}}{12}$$

$$=\frac{-13\pm\sqrt{25}}{12}$$

$$=\frac{-13\pm5}{12}$$

$$\boldsymbol{x=-\frac{2}{3},\ -\frac{3}{2}}$$

Factorisation

$(3x+2)(2x+3)=0$

$$\boldsymbol{x=-\frac{2}{3},\ -\frac{3}{2}}$$

(2) $15x^2+x-2=0$

Quadratic Formula

$$x=\frac{-1\pm\sqrt{1+120}}{30}$$

$$=\frac{-1\pm11}{30}$$

$$\boldsymbol{x=\frac{1}{3},\ -\frac{2}{5}}$$

Factorisation

$(5x+2)(3x-1)=0$

$$\boldsymbol{x=-\frac{2}{5},\ \frac{1}{3}}$$

J106b

(3) $4x^2-8x+3=0$

Quadratic Formula

$$x=\frac{4\pm\sqrt{16-12}}{4}$$

$$=\frac{4\pm2}{4}$$

$$\boldsymbol{x=\frac{3}{2},\ \frac{1}{2}}$$

Factorisation

$$(2x-3)(2x-1)=0$$

$$\boldsymbol{x=\frac{3}{2},\ \frac{1}{2}}$$

(4) $12x^2+4x-5=0$

Quadratic Formula

$$x=\frac{-2\pm\sqrt{4+60}}{12}$$

$$=\frac{-2\pm8}{12}$$

$$\boldsymbol{x=\frac{1}{2},\ -\frac{5}{6}}$$

Factorisation

$$(6x+5)(2x-1)=0$$

$$\boldsymbol{x=-\frac{5}{6},\ \frac{1}{2}}$$

J 107a KUMON

Quadratic Equations II

Factorise the following quadratic expressions using the quadratic formula.

Ex.

$6x^2-13x+6$

[Sol] Solving $6x^2-13x+6=0$

$$x=\frac{13\pm\sqrt{169-144}}{12}=\frac{13\pm5}{12}$$

$$x=\frac{3}{2},\ \frac{2}{3}$$

The quadratic equation with these solutions is:

$$6\left(x-\frac{3}{2}\right)\left(x-\frac{2}{3}\right)=0$$ ☜ Check that the coefficient of x^2 is 6.

$$2\left(x-\frac{3}{2}\right)\cdot3\left(x-\frac{2}{3}\right)=0$$ ☜ Break up 6 into 2 and 3.

$$(2x-3)(3x-2)=0$$

Therefore,

$$6x^2-13x+6=(2x-3)(3x-2)$$

(1) $12x^2+25x+12$

[Sol] Solving $12x^2+25x+12=0$

$$x=\frac{-25\pm\sqrt{625-576}}{24}=\frac{-25\pm7}{24}$$

$$x=-\frac{3}{4},\ -\frac{4}{3}$$

The quadratic equation with these solutions is:

$$12\left(x+\frac{3}{4}\right)\left(x+\frac{4}{3}\right)=0$$

$$4\left(x+\frac{3}{4}\right)\cdot3\left(x+\frac{4}{3}\right)=0$$

$$(4x+3)(3x+4)=0$$

Therefore,

$$12x^2+25x+12=\mathbf{(4x+3)(3x+4)}$$

J107b

（2） $15x^2-2x-24$

[Sol] Solving $15x^2-2x-24=0$

$$x=\frac{1\pm\sqrt{1+360}}{15}=\frac{1\pm19}{15}$$

$$x=\frac{4}{3},\ -\frac{6}{5}$$

The quadratic equation with these solutions is:

$$15\left(x-\frac{4}{3}\right)\left(x+\frac{6}{5}\right)=0$$

$$3\left(x-\frac{4}{3}\right)\cdot5\left(x+\frac{6}{5}\right)=0$$

$$(3x-4)(5x+6)=0$$

Therefore,

$$15x^2-2x-24=\mathbf{(3x-4)(5x+6)}$$

（3） $3x^2+7x-48$

[Sol] Solving $3x^2+7x-48=0$

$$x=\frac{-7\pm\sqrt{49+576}}{6}=\frac{-7\pm25}{6}$$

$$x=3,\ -\frac{16}{3}$$

The quadratic equation with these solutions is:

$$3(x-3)\left(x+\frac{16}{3}\right)=0$$

$$(x-3)(3x+16)=0$$

Therefore,

$$3x^2+7x-48=\mathbf{(x-3)(3x+16)}$$

J 108a KUMON

Quadratic Equations II

Factorise the following quadratic expressions using the quadratic formula.

(1) $x^2-\sqrt{2}\,x-12$

[Sol] Solving $x^2-\sqrt{2}\,x-12=0$

$$x=\frac{\sqrt{2}\pm\sqrt{2+48}}{2}=\frac{\sqrt{2}\pm\sqrt{50}}{2}=\frac{\sqrt{2}\pm 5\sqrt{2}}{2}$$

$$x=3\sqrt{2},\ -2\sqrt{2}$$

Therefore,

$$x^2-\sqrt{2}\,x-12=\boldsymbol{(x-3\sqrt{2})(x+2\sqrt{2})}$$

(2) $x^2-(2\sqrt{3}+1)x+2\sqrt{3}$

[Sol] Solving $x^2-(2\sqrt{3}+1)x+2\sqrt{3}=0$

$$x=\frac{(2\sqrt{3}+1)\pm\sqrt{(2\sqrt{3}+1)^2-4\times 2\sqrt{3}}}{2}$$

$$=\frac{(2\sqrt{3}+1)\pm\sqrt{12+4\sqrt{3}+1-8\sqrt{3}}}{2}$$

$$=\frac{(2\sqrt{3}+1)\pm\sqrt{13-4\sqrt{3}}}{2}$$

$$=\frac{(2\sqrt{3}+1)\pm\sqrt{13-2\sqrt{12}}}{2}$$

$$=\frac{(2\sqrt{3}+1)\pm\sqrt{(\sqrt{12}-1)^2}}{2}$$

$$=\frac{(2\sqrt{3}+1)\pm(2\sqrt{3}-1)}{2}$$

$$x=2\sqrt{3},\ 1$$

Therefore,

$$x^2-(2\sqrt{3}+1)x+2\sqrt{3}=\boldsymbol{(x-2\sqrt{3})(x-1)}$$

J108b

(3) $x^2-(a-2b)x-2ab$

[Sol] Solving $x^2-(a-2b)x-2ab=0$

$$x=\frac{(a-2b)\pm\sqrt{(a-2b)^2+4\times 2ab}}{2}$$

$$=\frac{(a-2b)\pm\sqrt{a^2-4ab+4b^2+8ab}}{2}$$

$$=\frac{(a-2b)\pm\sqrt{a^2+4ab+4b^2}}{2}$$

$$=\frac{(a-2b)\pm\sqrt{(a+2b)^2}}{2}$$

$$=\frac{(a-2b)\pm(a+2b)}{2}$$

$$x=a,\ -2b$$

Therefore,

$$x^2-(a-2b)x-2ab=\boldsymbol{(x-a)(x+2b)}$$

(4) $x^2-2ax+a^2-b^2$

[Sol] Solving $x^2-2ax+a^2-b^2=0$

$$x=a\pm\sqrt{a^2-(a^2-b^2)}$$

$$=a\pm\sqrt{b^2}$$

$$x=a\pm b$$

Therefore,

$$x^2-2ax+a^2-b^2=\boldsymbol{(x-a-b)(x-a+b)}$$

J 109a KUMON

Quadratic Equations II

Ex.

Given that one root of $x^2+bx+c=0$ is $1+\sqrt{2}$, find the values of b and c, and the other root. (Assume b and c are integers.)

(A root is another word for the solution of an equation.)

[Sol] $(1+\sqrt{2})^2+b(1+\sqrt{2})+c=0$ ☞ Substitute $x=1+\sqrt{2}$ into the quadratic equation.

$3+2\sqrt{2}+b+\sqrt{2}b+c=0$

Group terms with a factor $\sqrt{2}$.

$(3+b+c)+(2+b)\sqrt{2}=0$

We can deduce that,

This equation holds when $3+b+c=0$ and $2+b=0$.

$$\begin{cases} 3+b+c=0 & \cdots ① \\ 2+b=0 & \cdots ② \end{cases}$$

From ① and ②, $b=-2$ and $c=-1$

Thus $x^2-2x-1=0$ ☞ Put $b=-2$, $c=-1$ in the original equation.

and $x=1\pm\sqrt{2}$

Therefore, the other root is $x=1-\sqrt{2}$

1. Given that one root of $x^2+bx+c=0$ is $-1+\sqrt{2}$, find the values of b and c, and the other root. (Assume b and c are integers.)

[Sol] $(-1+\sqrt{2})^2+b(-1+\sqrt{2})+c=0$ ☞ Substitute $x=-1+\sqrt{2}$ into the quadratic equation.

$3-2\sqrt{2}-b+\sqrt{2}b+c=0$

$(3-b+c)+(b-2)\sqrt{2}=0$

We can deduce that,

$$\begin{cases} 3-b+c=0 & \cdots ① \\ b-2=0 & \cdots ② \end{cases}$$

From ① and ②, $\boldsymbol{b=2}$ and $\boldsymbol{c=-1}$

Thus $x^2+2x-1=0$

and $x=-1\pm\sqrt{2}$

Therefore, the other root is $\boldsymbol{x=-1-\sqrt{2}}$

J 109b

2. Given that one root of $ax^2+bx+2=0$ is $2-\sqrt{2}$, find the values of a and b, and the other root. (Assume a and b are integers.)

[Sol] $a(2-\sqrt{2})^2+b(2-\sqrt{2})+2=0$ ☞ Substitute $x=2-\sqrt{2}$ into the quadratic equation.

$a(6-4\sqrt{2})+b(2-\sqrt{2})+2=0$

$6a-4\sqrt{2}\,a+2b-\sqrt{2}\,b+2=0$

$(6a+2b+2)+(-4a-b)\sqrt{2}=0$

We can deduce that,

$$\begin{cases} 6a+2b+2=0 & \cdots ① \\ -4a-b=0 & \cdots ② \end{cases}$$

From ① and ②, $\boldsymbol{a=1}$ and $\boldsymbol{b=-4}$

Thus $x^2-4x+2=0$

and $x=2\pm\sqrt{2}$

Therefore, the other root is $\boldsymbol{x=2+\sqrt{2}}$

Note Summary

① • $x^2-6x+8=(x-4)(x-2)$
If the quadratic expression is set equal to zero, $x^2-6x+8=0$, there are two solutions: $x=4,\ 2$

• If we know that the roots of $x^2-6x+8=0$ are $x=4,\ 2$, we can use this to factorise $x^2-6x+8=(x-4)(x-2)$.

② If a, b and c are rational and the quadratic equation $ax^2+bx+c=0$ has an irrational solution $x=\alpha+\sqrt{\beta}$, then the other solution is $x=\alpha-\sqrt{\beta}$.

J 110a KUMON

Quadratic Equations II

1. Solve the following quadratic equations.

(1) $7x^2+32x-15=0$

$(7x-3)(x+5)=0$

$\boldsymbol{x=\frac{3}{7},\ -5}$

(2) $x^2-x-3=0$

$x=\frac{1\pm\sqrt{1+12}}{2}$

$\boldsymbol{=\frac{1\pm\sqrt{13}}{2}}$

(3) $\frac{(x-1)^2}{3}-\frac{x^2-1}{2}-1=0$

$2(x-1)^2-3(x^2-1)-6=0$

$-x^2-4x-1=0$

$x^2+4x+1=0$

$x=-2\pm\sqrt{4-1}$

$\boldsymbol{=-2\pm\sqrt{3}}$

(4) $0.2x^2-0.5x-1.2=0$

$2x^2-5x-12=0$

$(2x+3)(x-4)=0$

$\boldsymbol{x=-\frac{3}{2},\ 4}$

(5) $(x-4)(2x-3)=12$

$2x^2-11x+12-12=0$

$2x^2-11x=0$

$x(2x-11)=0$

$\boldsymbol{x=0,\ \frac{11}{2}}$

(6) $\frac{2}{3}x+\frac{1}{2}=\frac{1}{6}x^2$

$4x+3=x^2$

$x^2-4x-3=0$

$x=2\pm\sqrt{4+3}$

$\boldsymbol{=2\pm\sqrt{7}}$

2. Given that one root of $x^2-3x+a=0$ is $x=6$, find the value of a and the other root.

[Sol] $36-18+a=0$ ☜ Substitute $x=6$ into the quadratic equation.

$$18+a=0$$
$$a=-18$$

Therefore,

$$x^2-3x-18=0$$
$$(x-6)(x+3)=0$$
$$x=6,\ -3$$

Thus, $\boldsymbol{a=-18}$ and the other root is $\boldsymbol{x=-3}$

3. Given that one root of $x^2+6x+a=0$ is $x=-3+\sqrt{17}$, find the value of a and the other root.

[Sol] $(-3+\sqrt{17})^2+6(-3+\sqrt{17})+a=0$ ☜ Substitute $x=-3+\sqrt{17}$ into the quadratic equation.

$$26-6\sqrt{17}-18+6\sqrt{17}+a=0$$
$$8+a=0$$
$$a=-8$$

Therefore,

$$x^2+6x-8=0$$
$$x=-3\pm\sqrt{9+8}$$
$$=-3\pm\sqrt{17}$$

Thus, $\boldsymbol{a=-8}$ and the other root is $\boldsymbol{x=-3-\sqrt{17}}$

J111a KUMON

Quadratic Equations and Complex Numbers

1. Calculate the following expressions.

(1) $2^2 = \mathbf{4}$

(2) $5^2 = \mathbf{25}$

(3) $\left(\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$

(4) $(\sqrt{2})^2 = \mathbf{2}$

(5) $(\sqrt{3})^2 = \mathbf{3}$

(6) $(-2)^2 = \mathbf{4}$

(7) $(-5)^2 = \mathbf{25}$

(8) $\left(-\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$

(9) $(-\sqrt{2})^2 = \mathbf{2}$

(10) $(-\sqrt{3})^2 = \mathbf{3}$

Positive numbers, negative numbers and zero, form the set of ***real numbers***.
All positive and negative numbers give positive answers when squared. Zero gives zero when squared.

To express numbers that give negative answers when squared, we define the "***non-real***" ***number*** i. (This is called an ***imaginary number***.)

$$\boxed{i^2 = -1 \quad (\sqrt{-1} = i)}$$

2. Express the following square roots in terms of i.

Ex.

$\sqrt{-5}=\sqrt{5}\,i,\ \sqrt{-4}=\sqrt{4}\,i=2i,\ \sqrt{-8}=\sqrt{8}\,i=2\sqrt{2}\,i$

(1) $\sqrt{-3}=\boldsymbol{\sqrt{3}\,i}$

(2) $\sqrt{-6}=\boldsymbol{\sqrt{6}\,i}$

(3) $\sqrt{-9}=\sqrt{9}\,i=\boldsymbol{3i}$

(4) $\sqrt{-12}=\sqrt{12}i=\boldsymbol{2\sqrt{3}\,i}$

(5) $\sqrt{-18}=\sqrt{18}i=\boldsymbol{3\sqrt{2}\,i}$

(6) $\sqrt{-\frac{1}{25}}=\sqrt{\frac{1}{25}}\,i=\boldsymbol{\frac{1}{5}i}$

(7) $\sqrt{-\frac{3}{16}}=\sqrt{\frac{3}{16}}\,i=\boldsymbol{\frac{\sqrt{3}}{4}i}$

When $a \geqslant 0$, $\sqrt{-a}=\sqrt{a}\,i$

J112a KUMON

Quadratic Equations and Complex Numbers

Using $i^2 = -1$, calculate the following expressions.

Ex.

$(2i)^2 = 2i \times 2i = 2^2 \cdot i^2 = 4 \times (-1) = -4$

$(-2i)^2 = (-2)^2 \cdot i^2 = 4 \times (-1) = -4$

$(\sqrt{2}\, i)^2 = (\sqrt{2}\,)^2 \cdot i^2 = 2 \times (-1) = -2$

(1) $(3i)^2 = 3i \times 3i = 3^2 \cdot i^2 = 9 \times (-1) = \mathbf{-9}$

(2) $(-3i)^2 = (-3)^2 \cdot i^2 = 9 \times (-1) = \mathbf{-9}$

(3) $(4i)^2 = 16 \times (-1) = \mathbf{-16}$

(4) $(-7i)^2 = 49 \times (-1) = \mathbf{-49}$

(5) $(\sqrt{3}\, i)^2 = (\sqrt{3}\,)^2 \cdot i^2 = 3 \times (-1) = \mathbf{-3}$

(6) $(-\sqrt{3}\, i)^2 = (-\sqrt{3}\,)^2 \cdot i^2 = 3 \times (-1) = \mathbf{-3}$

(7) $(\sqrt{5}\, i)^2 = 5 \times (-1) = \mathbf{-5}$

(8) $(2\sqrt{2}\, i)^2 = 8 \times (-1) = \mathbf{-8}$

J112b

Ex.
$3i \times 2i = 6i^2 = -6$ ☜ Substitute $i^2 = -1$.

(9) $3i \times 4i = 12i^2 = \mathbf{-12}$

(10) $3i \times (-4i) = -12i^2 = \mathbf{12}$

(11) $\sqrt{3}\,i \times 2i = 2\sqrt{3}\,i^2 = \mathbf{-2\sqrt{3}}$

(12) $(-\sqrt{3}\,i) \times 4i = -4\sqrt{3}\,i^2 = \mathbf{4\sqrt{3}}$

Ex.
$\sqrt{-4} \times \sqrt{-25} = 2i \times 5i = 10i^2 = -10$

(13) $\sqrt{-9} \times \sqrt{-16} = 3i \times 4i = 12i^2 = \mathbf{-12}$

(14) $2\sqrt{-2} \times 3\sqrt{-3} = 2\sqrt{2}\,i \times 3\sqrt{3}\,i = 6\sqrt{6}\,i^2 = \mathbf{-6\sqrt{6}}$

(15) $\sqrt{-4} \times \sqrt{-9} = 2i \times 3i = 6i^2 = \mathbf{-6}$

(16) $\sqrt{(-4) \times (-9)} = \sqrt{36} = \mathbf{6}$

Conclusion:
$\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ unless $a < 0$ and $b < 0$

Note the answers to (15) and (16) are not the same.

J 1 1 3a KUMON

Quadratic Equations and Complex Numbers

Calculate the following expressions.

Ex.

$\sqrt{-4} \div \sqrt{16} = \dfrac{2i}{4} = \dfrac{1}{2}i$

$\sqrt{4} \div \sqrt{-16} = \dfrac{2}{4i} = \dfrac{2i}{4i^2} = \dfrac{2i}{-4} = -\dfrac{1}{2}i$ ☜ To eliminate i from the denominator, multiply the numerator and denominator of $\dfrac{2}{4i}$ by i.

(1) $\sqrt{-9} \div \sqrt{16} = \dfrac{3i}{4} = \boldsymbol{\dfrac{3}{4}i}$

(2) $\sqrt{9} \div \sqrt{-16} = \dfrac{3}{4i} = \dfrac{3i}{4i^2} = \dfrac{3i}{-4} = \boldsymbol{-\dfrac{3}{4}i}$

(3) $\sqrt{-9} \div \sqrt{25} = \dfrac{3i}{5} = \boldsymbol{\dfrac{3}{5}i}$

(4) $\sqrt{9} \div \sqrt{-25} = \dfrac{3}{5i} = \dfrac{3i}{5i^2} = \dfrac{3i}{-5} = \boldsymbol{-\dfrac{3}{5}i}$

(5) $\dfrac{\sqrt{9}}{\sqrt{-25}} = \dfrac{3}{5i} = \dfrac{3i}{5i^2} = \dfrac{3i}{-5} = \boldsymbol{-\dfrac{3}{5}i}$

(6) $\sqrt{\dfrac{9}{-25}} = \sqrt{-\dfrac{9}{25}} = \sqrt{\dfrac{9}{25}}\,i = \boldsymbol{\dfrac{3}{5}i}$

Conclusion:

$\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ <u>unless</u> $a > 0$ and $b < 0$

Note the answers to (5) and (6) are <u>not</u> the same.

J113b

Ex.

$(2+i)(3+2i)=6+7i+2i^2=4+7i$

(7) $(2-i)(3+2i)=6+i-2i^2=\mathbf{8+\boldsymbol{i}}$

(8) $(3-2i)(1+3i)=3+7i-6i^2=\mathbf{9+7\boldsymbol{i}}$

(9) $(2+\sqrt{2}\,i)(3+\sqrt{2}\,i)=6+5\sqrt{2}\,i+2i^2=\mathbf{4+5\sqrt{2}\,\boldsymbol{i}}$

(10) $(5+\sqrt{3}\,i)(4-\sqrt{3}\,i)=20-\sqrt{3}\,i-3i^2=\mathbf{23-\sqrt{3}\,\boldsymbol{i}}$

(11) $(3+2i)(3-2i)=9-4i^2=\mathbf{13}$

(12) $(3+2i)^2=9+12i+4i^2=\mathbf{5+12\boldsymbol{i}}$

$3+2i$ and $4-3i$ are called ***complex numbers***.

- $a+bi$ is the standard form of a ***complex number***, (a and b are real numbers).
- When $b=0$, $a+bi$ is a ***real number***, (since $a+0\cdot i=a$).
- When $a=0$, $a+bi$ is an ***imaginary number***.

J114a KUMON

Quadratic Equations and Complex Numbers

Calculate the following expressions.

(1) $(2+\sqrt{-2})(3+\sqrt{-2})=6+5\sqrt{2}\,i+2i^2=\mathbf{4+5\sqrt{2}\,\boldsymbol{i}}$

(2) $(2-5\sqrt{2}\,i)(3+4\sqrt{2}\,i)=6-7\sqrt{2}\,i-40i^2=\mathbf{46-7\sqrt{2}\,\boldsymbol{i}}$

(3) $(4-\sqrt{-2})(4+\sqrt{-2})=16-2i^2=\mathbf{18}$

(4) $(\sqrt{3}\,i+2)^2=3i^2+4\sqrt{3}\,i+4=\mathbf{1+4\sqrt{3}\,\boldsymbol{i}}$

(5) $(\sqrt{-5}+1)^2=(\sqrt{5}\,i+1)^2=5i^2+2\sqrt{5}\,i+1=\mathbf{-4+2\sqrt{5}\,\boldsymbol{i}}$

(6) $(-3-2\sqrt{-2})^2=(-3-2\sqrt{2}\,i)^2=9+12\sqrt{2}\,i+8i^2$
$=\mathbf{1+12\sqrt{2}\boldsymbol{i}}$

J114b

Ex.

$$\frac{3-i}{3+2i}=\frac{(3-i)(3-2i)}{(3+2i)(3-2i)}=\frac{9-9i+2i^2}{9+4}=\frac{7-9i}{13}$$

Multiply the numerator and denominator of $\frac{3-i}{3+2i}$ by $(3-2i)$.

(7) $\frac{1+i}{3+2i}=\frac{(1+i)(3-2i)}{(3+2i)(3-2i)}=\frac{3+i-2i^2}{9+4}=\boldsymbol{\frac{5+i}{13}}$

(8) $\frac{5-i}{2-3i}=\frac{(5-i)(2+3i)}{(2-3i)(2+3i)}=\frac{10+13i-3i^2}{4+9}=\frac{13+13i}{13}=\boldsymbol{1+i}$

(9) $\frac{1+i}{1-i}=\frac{(1+i)^2}{(1-i)(1+i)}=\frac{1+2i+i^2}{1+1}=\frac{2i}{2}=\boldsymbol{i}$

(10) $\frac{2-i}{2+i}=\frac{(2-i)^2}{(2+i)(2-i)}=\frac{4-4i+i^2}{4+1}=\boldsymbol{\frac{3-4i}{5}}$

(11) $\frac{1}{2-i}+\frac{1}{2+i}=\frac{2+i+2-i}{(2-i)(2+i)}=\frac{4}{4+1}=\boldsymbol{\frac{4}{5}}$

(12) $\frac{i}{2-i}-\frac{i}{2+i}=\frac{i(2+i)-i(2-i)}{(2-i)(2+i)}=\frac{2i+i^2-2i+i^2}{4+1}=\boldsymbol{-\frac{2}{5}}$

J115a KUMON

Quadratic Equations and Complex Numbers

1. Calculate the following expressions.

Ex.

$i^3 = i^2 \cdot i = (-1) \cdot i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

(1) $i^5 = (i^2)^2 \cdot i = (-1)^2 \cdot i = \boldsymbol{i}$

(2) $i^6 = (i^2)^3 = (-1)^3 = \mathbf{-1}$

(3) $i^7 = (i^2)^3 \cdot i = (-1)^3 \cdot i = \boldsymbol{-i}$

(4) $i^8 = (i^2)^4 = (-1)^4 = \mathbf{1}$

(5) $\dfrac{1}{i} = \dfrac{i}{i \cdot i} = \dfrac{i}{-1} = \boldsymbol{-i}$

(6) $\dfrac{1}{i^2} = \dfrac{1}{-1} = \mathbf{-1}$

(7) $\dfrac{1}{i^3} = \dfrac{1}{i^2 \cdot i} = -\dfrac{1}{i} = -\dfrac{i}{i^2} = -\dfrac{i}{-1} = \boldsymbol{i}$

(8) $\dfrac{1}{i^4} = \dfrac{1}{(i^2)^2} = \dfrac{1}{(-1)^2} = \mathbf{1}$

2. Solve for the real numbers x and y.

Ex.

$(x+y)+(x-y)i=7+3i$

This equation holds when ① $x+y=7$ and ② $x-y=3$.

[Sol] Comparing real and imaginary parts on each side:

$$\begin{cases} x+y=7 & \cdots ① \\ x-y=3 & \cdots ② \end{cases}$$

From ① and ②,

$x=5, \ y=2$

(1) $(2x-y)+(y-x)i=1+3i$

Comparing real and imaginary parts on each side:

$$\begin{cases} 2x-y=1 & \cdots ① \\ y-x=3 & \cdots ② \end{cases}$$

From ① and ②,

$\boldsymbol{x=4, \ y=7}$

First, group together real and imaginary parts.

(2) $(1+i)x-(1-2i)y=1+4i$

[Sol] $(\ \boldsymbol{x-y}\ )+(\ \boldsymbol{x+2y}\ )i=1+4i$

Comparing real and imaginary parts on each side:

$$\begin{cases} x-y=1 & \cdots ① \\ x+2y=4 & \cdots ② \end{cases}$$

From ① and ②,

$\boldsymbol{x=2, \ y=1}$

Two complex numbers, $a+bi$ and $c+di$, are equal when they have identical real parts and identical imaginary parts.

$$a+bi=c+di \Leftrightarrow a=c, \ b=d$$

Also $$a+bi=0 \Leftrightarrow a=0, \ b=0$$

$\Leftrightarrow$ means "if and only if"

J116a KUMON

Quadratic Equations and Complex Numbers

Quadratic Formula I

When $ax^2+bx+c=0$,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Quadratic Formula II

When $ax^2+2b'x+c=0$,

$$x=\frac{-b'\pm\sqrt{b'^2-ac}}{a}$$

Ex.

$3x^2+5x-7=0$

$$x=\frac{-5\pm\sqrt{25-4\cdot3\cdot(-7)}}{6}$$

$$=\frac{-5\pm\sqrt{109}}{6}$$

Ex.

$5x^2-8x+4=0$

$$x=\frac{4\pm\sqrt{16-20}}{5}$$

$$=\frac{4\pm\sqrt{-4}}{5}$$

Substitute $\sqrt{-4}=2i$.

$$=\frac{4\pm2i}{5}$$

Solve the following equations by using the quadratic formula.

(1) $2x^2+5x+4=0$

$$\boldsymbol{x}=\frac{-5\pm\sqrt{25-4\cdot2\cdot4}}{4}$$

$$=\frac{-5\pm\sqrt{-7}}{4}$$

$$=\boldsymbol{\frac{-5\pm\sqrt{7}\,i}{4}}$$

(2) $4x^2-4x+5=0$

$$\boldsymbol{x}=\frac{2\pm\sqrt{4-20}}{4}$$

$$=\frac{2\pm\sqrt{-16}}{4}$$

$$=\frac{2\pm4i}{4}$$

$$=\boldsymbol{\frac{1\pm2i}{2}}$$

When the solution of an equation is a complex number, we call it a ***complex root*** or ***complex solution***.

J116b

(3) $3x^2-8x+7=0$

$$\boldsymbol{x}=\frac{4\pm\sqrt{16-21}}{3}$$

$$=\frac{4\pm\sqrt{-5}}{3}$$

$$\boldsymbol{=\frac{4\pm\sqrt{5}\,i}{3}}$$

(6) $3x^2-12=0$

$$\boldsymbol{x}=\frac{0\pm\sqrt{0-3\cdot(-12)}}{3}$$

$$=\pm\frac{6}{3}$$

$$\boldsymbol{=\pm 2}$$

(4) $2x^2+8x+9=0$

$$\boldsymbol{x}=\frac{-4\pm\sqrt{16-18}}{2}$$

$$=\frac{-4\pm\sqrt{-2}}{2}$$

$$\boldsymbol{=\frac{-4\pm\sqrt{2}\,i}{2}}$$

(7) $2x^2=5x$

$$2x^2-5x=0$$

$$x=\frac{5\pm\sqrt{25-4\cdot 2\cdot 0}}{4}$$

$$=\frac{5\pm 5}{4}$$

$$\boldsymbol{x=\frac{5}{2},\ 0}$$

(5) $4x^2-12x+9=0$

$$\boldsymbol{x}=\frac{6\pm\sqrt{36-36}}{4}$$

$$=\frac{6\pm 0}{4}$$

$$\boldsymbol{=\frac{3}{2}}$$

(8) $-x^2+4x-5=0$

$$x^2-4x+5=0$$

$$\boldsymbol{x}=\frac{2\pm\sqrt{4-5}}{1}$$

$$=2\pm\sqrt{-1}$$

$$\boldsymbol{=2\pm i}$$

J117a KUMON

Quadratic Equations and Complex Numbers

Solve the following quadratic equations.

(1) $5x^2-6x+4=0$

$$\boldsymbol{x}=\frac{3\pm\sqrt{9-20}}{5}$$

$$=\boldsymbol{\frac{3\pm\sqrt{11}\,i}{5}}$$

(2) $9x^2+12x+4=0$

$(3x+2)^2=0$

$$\boldsymbol{x=-\frac{2}{3}}$$

(3) $2x^2+5x+5=0$

$$\boldsymbol{x}=\frac{-5\pm\sqrt{25-40}}{4}$$

$$=\boldsymbol{\frac{-5\pm\sqrt{15}\,i}{4}}$$

(4) $-x^2+3x-5=0$

$x^2-3x+5=0$

$$\boldsymbol{x}=\frac{3\pm\sqrt{9-20}}{2}$$

$$=\boldsymbol{\frac{3\pm\sqrt{11}\,i}{2}}$$

(5) $-3x^2+8x-7=0$

$3x^2-8x+7=0$

$$\boldsymbol{x}=\frac{4\pm\sqrt{16-21}}{3}$$

$$=\boldsymbol{\frac{4\pm\sqrt{5}\,i}{3}}$$

J117b

(6) $2x(3-x)=4-3x$

$2x^2-9x+4=0$

$(2x-1)(x-4)=0$

$\boldsymbol{x=\frac{1}{2},\ 4}$

(8) $x^2-\sqrt{3}\,x-2=0$

$\boldsymbol{x}=\dfrac{\sqrt{3}\pm\sqrt{3+8}}{2}$

$\boldsymbol{=\dfrac{\sqrt{3}\pm\sqrt{11}}{2}}$

(7) $(x-5)(2x-1)=5$

$2x^2-11x+5=5$

$2x^2-11x=0$

$x(2x-11)=0$

$\boldsymbol{x=0,\ \frac{11}{2}}$

(9) $x^2+\sqrt{5}\,x-10=0$

$x=\dfrac{-\sqrt{5}\pm\sqrt{5+40}}{2}$

$=\dfrac{-\sqrt{5}\pm\sqrt{45}}{2}=\dfrac{-\sqrt{5}\pm3\sqrt{5}}{2}$

$\boldsymbol{x=\sqrt{5},\ -2\sqrt{5}}$

[Alternative Solution
$(x+2\sqrt{5})(x-\sqrt{5})=0$
$\boldsymbol{x=-2\sqrt{5},\ \sqrt{5}}$]

J118a KUMON

Quadratic Equations and Complex Numbers

Solve the following quadratic equations.

(1) $(x+1)^2=(x+2)^2+(x+3)^2$

$x^2+2x+1=x^2+4x+4+x^2+6x+9$

$x^2+8x+12=0$

$(x+6)(x+2)=0$

$\boldsymbol{x=-6,\ -2}$

(3) $x^2=\dfrac{2}{3}(x+1)$

$3x^2=2(x+1)$

$3x^2-2x-2=0$

$x=\dfrac{1\pm\sqrt{1+6}}{3}$

$\boldsymbol{=\dfrac{1\pm\sqrt{7}}{3}}$

(2) $\dfrac{1}{4}=(2+x)^2$

$1=4(2+x)^2$

$4x^2+16x+15=0$

$(2x+5)(2x+3)=0$

$\boldsymbol{x=-\dfrac{5}{2},\ -\dfrac{3}{2}}$

(4) $\dfrac{x(x+3)}{3}=\dfrac{5x-1}{4}$

$4x(x+3)=3(5x-1)$

$4x^2+12x=15x-3$

$4x^2-3x+3=0$

$x=\dfrac{3\pm\sqrt{9-48}}{8}$

$\boldsymbol{=\dfrac{3\pm\sqrt{39}\,i}{8}}$

J118b

(5) $0.3x^2-2x+3.6=0$

$3x^2-20x+36=0$

$\boldsymbol{x}=\dfrac{10\pm\sqrt{100-108}}{3}$

$=\boldsymbol{\dfrac{10\pm2\sqrt{2}\,i}{3}}$

(6) $0.2x^2-0.5x+0.4=0$

$2x^2-5x+4=0$

$\boldsymbol{x}=\dfrac{5\pm\sqrt{25-32}}{4}$

$=\boldsymbol{\dfrac{5\pm\sqrt{7}\,i}{4}}$

(7) $x^2-2mx+n^2=0$

$\boldsymbol{x=m\pm\sqrt{m^2-n^2}}$

(8) $x^2-4ax+3a^2=0$

$(x-3a)(x-a)=0$

$\boldsymbol{x=3a,\ a}$

(9) $x^2-2ax+a^2-b^2=0$

$[x-(a+b)][x-(a-b)]=0$

$\boldsymbol{x=a+b,\ a-b}$

〔$\boldsymbol{x=a\pm b}$〕

J 1 1 9a KUMON

Quadratic Equations and Complex Numbers

1. Factorise the following expressions using complex numbers.

Ex.

x^2-2x+3

[Sol] Solving $x^2-2x+3=0$

$$x=1\pm\sqrt{1-3}$$
$$=1\pm\sqrt{2}\,i$$

$$[x-(1+\sqrt{2}\,i)][x-(1-\sqrt{2}\,i)]=0$$

Thus, $x^2-2x+3=(x-1-\sqrt{2}\,i)(x-1+\sqrt{2}\,i)$

(1) x^2-2x+5

Solving $x^2-2x+5=0$

$$x=1\pm\sqrt{1-5}$$
$$=1\pm\sqrt{-4}$$
$$=1\pm 2i$$
$$x=1+2i,\ 1-2i$$

$$[x-(1+2i)][x-(1-2i)]=0$$
$$(x-1-2i)(x-1+2i)=0$$

Thus,

$$x^2-2x+5$$
$$=\boldsymbol{(x-1-2i)(x-1+2i)}$$

(2) x^2+1

Solving $x^2+1=0$

$$x^2=-1$$
$$x=\pm i$$

$$(x-i)(x+i)=0$$

Thus, $x^2+1=\boldsymbol{(x-i)(x+i)}$

Ex.

Given that one root of $ax^2-2x+b=0$ is $1+2i$, find the values of a and b, and the other root. (Assume a and b are real numbers.)

[Sol] Substitute $x=1+2i$ into the quadratic equation,

$$a(1+2i)^2-2(1+2i)+b=0$$
$$-3a+4ai-2-4i+b=0$$
$$-3a+b-2+(4a-4)i=0$$

Group together real and imaginary terms.

Therefore,

$$\begin{cases} -3a+b-2=0 & \cdots ① \\ 4a-4=0 & \cdots ② \end{cases}$$

From ① and ②, $a=1,\ b=5$

Substituting into $ax^2-2x+b=0$.

$$x^2-2x+5=0$$
$$x=1\pm 2i$$

Thus, the other root is $x=1-2i$

2. Given that one root of $x^2-ax+b=0$ is $1+i$, find the values of a and b, and the other root.

[Sol] Substitute $x=1+i$ in the quadratic equation,

$$(1+i)^2-a(1+i)+b=0$$
$$2i-a-ai+b=0$$
$$(-a+b)+(2-a)i=0$$

Therefore,

$$\begin{cases} -a+b=0 & \cdots ① \\ 2-a=0 & \cdots ② \end{cases}$$

From ① and ②, $\boldsymbol{a=2,\ b=2}$

Substituting into $x^2-ax+b=0$.

$$x^2-2x+2=0$$
$$x=1\pm i$$

Thus, the other root is $\boldsymbol{x=1-i}$

If the quadratic equation $ax^2+bx+c=0$ has a complex solution $x=\alpha+\beta i$, then the other one is $x=\alpha-\beta i$.

J I 20a KUMON

Quadratic Equations and Complex Numbers

1. Solve the following quadratic equations.

(1) $-x^2+12x+8=0$

$x^2-12x-8=0$

$\boldsymbol{x}=6\pm\sqrt{36+8}$

$\boldsymbol{=6\pm2\sqrt{11}}$

(2) $\frac{1}{9}=(2+x)^2$

$1=9(2+x)^2$

$9x^2+36x+35=0$

$(3x+7)(3x+5)=0$

$\boldsymbol{x=-\frac{7}{3},\ -\frac{5}{3}}$

(3) $0.5x^2+0.2x+0.1=0$

$5x^2+2x+1=0$

$\boldsymbol{x}=\frac{-1\pm\sqrt{1-5}}{5}$

$\boldsymbol{=\frac{-1\pm2i}{5}}$

(4) $x^2-2ax+2a^2=0$

$\boldsymbol{x}=a\pm\sqrt{a^2-2a^2}$

$=a\pm a\sqrt{-1}$

$\boldsymbol{=a\pm ai}$

(5) $x^2+2ax+(2a-1)=0$

$(x+1)[x+(2a-1)]=0$

$\boldsymbol{x=-1,\ -2a+1}$

J I20b

2. Calculate the following expressions.

（1） $\sqrt{-3}\sqrt{-5}=\sqrt{3}i\cdot\sqrt{5}i=\sqrt{15}i^2=\mathbf{-\sqrt{15}}$

（2） $\sqrt{(-3)\times(-5)}=\mathbf{\sqrt{15}}$

（3） $\dfrac{\sqrt{3}}{\sqrt{-5}}=\dfrac{\sqrt{3}}{\sqrt{5}i}=\dfrac{\sqrt{3}\cdot\sqrt{5}\cdot i}{\sqrt{5}\cdot\sqrt{5}\cdot i^2}=\dfrac{\sqrt{15}i}{-5}=\boldsymbol{-\dfrac{\sqrt{15}}{5}i}$

（4） $\sqrt{\dfrac{3}{-5}}=\sqrt{\dfrac{3}{5}}i=\boldsymbol{\dfrac{\sqrt{15}}{5}i}$

（5） $(3+2i)(5-6i)=15-8i-12i^2=\boldsymbol{27-8i}$

（6） $(3-\sqrt{-2})^2=(3-\sqrt{2}i)^2=9-6\sqrt{2}i+2i^2=\boldsymbol{7-6\sqrt{2}i}$

（7） $\dfrac{5-i}{2-3i}=\dfrac{(5-i)(2+3i)}{(2-3i)(2+3i)}=\dfrac{10+13i-3i^2}{4+9}=\dfrac{13+13i}{13}=\boldsymbol{1+i}$

J 121 a KUMON

Discriminant

Quadratic Formula I

When $ax^2+bx+c=0$,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

1. Solve the following equations by using the quadratic formula.

(1) $x^2+8x+15=0$

$$x=\frac{-8\pm\sqrt{64-60}}{2}$$
$$=-4\pm1$$
$$\boldsymbol{x=-3,\ -5}$$

(4) $x^2-4x+4=0$

$$\boldsymbol{x}=\frac{4\pm\sqrt{16-16}}{2}$$
$$\boldsymbol{=2}$$

(2) $x^2+8x+16=0$

$$\boldsymbol{x}=\frac{-8\pm\sqrt{64-64}}{2}$$
$$\boldsymbol{=-4}$$

(5) $x^2-4x+5=0$

$$\boldsymbol{x}=\frac{4\pm\sqrt{16-20}}{2}$$
$$\boldsymbol{=2\pm i}$$

(3) $x^2+8x+17=0$

$$\boldsymbol{x}=\frac{-8\pm\sqrt{64-68}}{2}$$
$$\boldsymbol{=-4\pm i}$$

(6) $x^2-4x+2=0$

$$\boldsymbol{x}=\frac{4\pm\sqrt{16-8}}{2}$$
$$\boldsymbol{=2\pm\sqrt{2}}$$

Note: It is also possible to solve these equations using Quadratic Formula II.

From the results above, there are three types of solutions:

- ➢ 2 different real solutions, (1) and **(6)**.
- ➢ a repeated real solution, (i.e. 1 distinct) (2) and **(4)**.
- ➢ 2 different complex solutions, (3) and **(5)**.

J121b

2. Solve the following equations by using the quadratic formula, then fill in the blank boxes $\boxed{}$.

(1) $x^2-5x+5=0$

$$x=\frac{5\pm\sqrt{25-20}}{2}$$

$$=\frac{\mathbf{5\pm\sqrt{5}}}{\mathbf{2}}$$

The quadratic equation $x^2-5x+5=0$ has 2 different $\boxed{\textbf{real}}$ solutions.

$b^2-4ac=5^2-4\cdot1\cdot5=5>0$

(2) $x^2-10x+25=0$

$$x=\frac{10\pm\sqrt{100-100}}{2}$$

$$=\mathbf{5}$$

The quadratic equation $x^2-10x+25=0$ has a $\boxed{\textbf{repeated}}$ real solution.

$b^2-4ac=\boxed{\mathbf{0}}$

(3) $x^2-5x+7=0$

$$x=\frac{5\pm\sqrt{25-28}}{2}$$

$$=\frac{\mathbf{5\pm\sqrt{3}\,\boldsymbol{i}}}{\mathbf{2}}$$

The quadratic equation $x^2-5x+7=0$ has 2 different $\boxed{\textbf{complex}}$ solutions.

$b^2-4ac=\boxed{\mathbf{-3}}<0$

Given a quadratic equation $ax^2+bx+c=0$, the solution is:

$$x=\frac{-b\pm\sqrt{\boldsymbol{b^2-4ac}}}{2a}$$

We can calculate $\boldsymbol{D=b^2-4ac}$ (the term inside the square root $\sqrt{}$) to determine what types of solutions a quadratic equation may have.

i) When $D>0$, there are 2 different real solutions, e.g. (1).

ii) When $D=0$, there is a repeated real solution, e.g. (2).

iii) When $D<0$, there are 2 different complex solutions, e.g. (3).

* $D=b^2-4ac$ is called the ***Discriminant***.

J 122a KUMON

Discriminant

For the quadratic equation $ax^2+bx+c=0$, the discriminant is $D=b^2-4ac$.

When $D>0$ $\Leftrightarrow$ There are 2 different real solutions.

When $D=0$ $\Leftrightarrow$ There is a repeated real solution.

When $D<0$ $\Leftrightarrow$ There are 2 different complex solutions.

In each question, calculate the discriminant, then state what type of solution(s) the given equation has.

Ex.

$x^2+5x+3=0$

[Sol] $D=5^2-4\cdot1\cdot3=13>0$

Therefore, there are 2 different real solutions.

(1) $x^2+5x-1=0$

$D=5^2-4\cdot1\cdot(-1)$
$=29>0$

Therefore, **there are 2 different real solutions.**

(2) $3x^2+3x+1=0$

$D=3^2-4\cdot3\cdot1$
$=-3<0$

Therefore, **there are 2 different complex solutions.**

(3) $4x^2-12x+9=0$

$D=(-12)^2-4\cdot4\cdot9$
$=0$

Therefore, **there is a repeated real solution.**

(4) $3x^2+7x+4=0$

$D=7^2-4\cdot3\cdot4$
$=1>0$

Therefore, **there are 2 different real solutions.**

Note: To "discriminate the solutions" means to find whether the quadratic equation has 2 different real solutions, 1 repeated real solution, or 2 different complex solutions.

J I22b

(5) $2x^2+6x+5=0$

$$D=6^2-4\cdot2\cdot5$$
$$=-4<0$$

Therefore, **there are 2 different complex solutions.**

(6) $5x^2-10x+5=0$

$$D=(-10)^2-4\cdot5\cdot5$$
$$=0$$

Therefore, **there is a repeated real solution.**

(7) $(x+3)^2+7=0$

$$x^2+6x+16=0$$
$$D=6^2-4\cdot1\cdot16$$
$$=-28<0$$

Therefore, **there are 2 different complex solutions.**

(8) $2x^2+\sqrt{3}x+3=0$

$$D=(\sqrt{3})^2-4\cdot2\cdot3$$
$$=-21<0$$

Therefore, **there are 2 different complex solutions.**

(9) $x^2+2\sqrt{2}x+2=0$

$$D=(2\sqrt{2})^2-4\cdot1\cdot2$$
$$=0$$

Therefore, **there is a repeated real solution.**

(10) $\sqrt{2}x^2-3x+2\sqrt{2}=0$

$$D=(-3)^2-4\cdot\sqrt{2}\cdot2\sqrt{2}$$
$$=-7<0$$

Therefore, **there are 2 different complex solutions.**

J I23a KUMON

Discriminant

For a quadratic equation $ax^2+2b'x+c=0$ (where the coefficient of x is an even number), we can discriminate solutions using:

$$\frac{D}{4}=b'^2-ac$$

Discriminate the solutions of the following equations.

Ex.

$x^2+4x+5=0$ ☞ $x^2+2\cdot 2x+5=0$

[Sol] $\frac{D}{4}=2^2-1\cdot 5=-1<0$

Therefore, there are 2 different complex solutions.

(1) $25x^2+10x+1=0$

[Sol] $\frac{D}{4}=5^2-25\cdot 1=0$

Therefore, **there is a repeated real solution.**

(2) $2x^2+6x+3=0$

$\frac{D}{4}=3^2-2\cdot 3=3>0$

Therefore, **there are 2 different real solutions.**

(3) $25x^2-30x+9=0$

$\frac{D}{4}=(-15)^2-25\cdot 9=0$

Therefore, **there is a repeated real solution.**

(4) $x^2-5x+3=0$

$D=(-5)^2-4\cdot 1\cdot 3=13>0$

Therefore, **there are 2 different real solutions.**

(5) $2x^2-4x+3=0$

$$\frac{D}{4}=(-2)^2-2\cdot3=-2<0$$

Therefore, **there are 2 different complex solutions.**

(6) $x^2-3x-2=0$

$$D=(-3)^2-4\cdot1\cdot(-2)=17>0$$

Therefore, **there are 2 different real solutions.**

(7) $3x^2-7x+6=0$

$$D=(-7)^2-4\cdot3\cdot6=-23<0$$

Therefore, **there are 2 different complex solutions.**

(8) $3x^2-12x+13=0$

$$\frac{D}{4}=(-6)^2-3\cdot13=-3<0$$

Therefore, **there are 2 different complex solutions.**

(9) $x^2+2\sqrt{3}\,x+3=0$

$$\frac{D}{4}=(\sqrt{3})^2-1\cdot3=0$$

Therefore, **there is a repeated real solution.**

(10) $2x^2-2\sqrt{3}\,x+3=0$

$$\frac{D}{4}=(-\sqrt{3})^2-2\cdot3=-3<0$$

Therefore, **there are 2 different complex solutions.**

J I24a KUMON

Discriminant

Determine the value of k for which each equation has a ***repeated real solution***.

For $ax^2+bx+c=0$,
the discriminant is

$$D=b^2-4ac$$

For $ax^2+2b'x+c=0$,
use

$$\frac{D}{4}=b'^2-ac$$

Ex.

$2x^2-5x+k=0$

[Sol] $D=25-8k=0$

For a ***repeated real solution***, the discriminant must equal zero, $D=0$.

Therefore, $k=\dfrac{25}{8}$

(1) $3x^2-5x+k=0$

$D=25-12k=0$

Therefore, $\boldsymbol{k=\dfrac{25}{12}}$

(2) $3x^2-8x+k=0$

$\dfrac{D}{4}=16-3k=0$

Therefore, $\boldsymbol{k=\dfrac{16}{3}}$

(3) $2x^2+kx+k-2=0$

$$\begin{aligned} D&=k^2-8(k-2)\\ &=k^2-8k+16\\ &=(k-4)^2=0 \end{aligned}$$

Therefore, $\boldsymbol{k=4}$

(4) $3x^2+2kx+(2k-3)=0$

$$\begin{aligned} \frac{D}{4}&=k^2-3(2k-3)\\ &=k^2-6k+9\\ &=(k-3)^2=0 \end{aligned}$$

Therefore, $\boldsymbol{k=3}$

J124b

(5) $x^2+2(k+1)x+2(1+k^2)=0$

$$\begin{aligned}\frac{D}{4}&=(k+1)^2-2(1+k^2)\\&=k^2+2k+1-2-2k^2\\&=-k^2+2k-1=0\end{aligned}$$

$k^2-2k+1=0$

$(k-1)^2=0$

Therefore, $\boldsymbol{k=1}$

(7) $kx^2-x-(k-2)=0$

$$\begin{aligned}D&=1+4k(k-2)\\&=4k^2-8k+1=0\end{aligned}$$

Therefore,

$$\begin{aligned}\boldsymbol{k}&=\frac{4\pm\sqrt{16-4}}{4}\\&=\frac{4\pm2\sqrt{3}}{4}\\&=\boldsymbol{\frac{2\pm\sqrt{3}}{2}}\end{aligned}$$

(6) $3x^2+4kx+4k=0$

$$\begin{aligned}\frac{D}{4}&=4k^2-12k\\&=4k(k-3)=0\end{aligned}$$

Therefore, $\boldsymbol{k=0,\ 3}$

(8) $kx^2+(k+1)x-(k+2)=0$

$$\begin{aligned}D&=(k+1)^2+4k(k+2)\\&=k^2+2k+1+4k^2+8k\\&=5k^2+10k+1=0\end{aligned}$$

Therefore,

$$\begin{aligned}\boldsymbol{k}&=\frac{-5\pm\sqrt{25-5}}{5}\\&=\boldsymbol{\frac{-5\pm2\sqrt{5}}{5}}\end{aligned}$$

J 125a

Discriminant

Determine the value of a for which each equation has ***real solutions***; $(D \geqslant 0)$.

Ex.

$x^2-3x+2a=0$

[Sol] $D=9-8a \geqslant 0$

Therefore, $a \leqslant \frac{9}{8}$

A quadratic equation has 2 real solutions when $D>0$. A quadratic equation has 1 repeated real solution when $D=0$. For the equation to have ***real solutions***, the discriminant must be greater or equal to zero, $D \geqslant 0$.

(1) $x^2-4x+3a=0$

[Sol] $\frac{D}{4}=4-3a \geqslant 0$

Therefore, $\boldsymbol{a \leqslant \frac{4}{3}}$

(2) $3x^2-7x+2a-1=0$

[Sol] $D=49-12(2a-1)$

$=-24a+61 \geqslant 0$

Therefore, $\boldsymbol{a \leqslant \frac{61}{24}}$

(3) $2x^2-4x+a+2=0$

$\frac{D}{4}=4-2(a+2)$

$=-2a \geqslant 0$

Therefore, $\boldsymbol{a \leqslant 0}$

(4) $3x^2-3x+1-a=0$

$D=9-12(1-a)$

$=12a-3 \geqslant 0$

Therefore, $\boldsymbol{a \geqslant \frac{1}{4}}$

J125b

(5) $4x^2-4(a-3)x+a^2=0$

$$\frac{D}{4}=4(a-3)^2-4a^2$$
$$=4[(a-3)^2-a^2]$$
$$=4(a^2-6a+9-a^2)$$
$$=4(-6a+9)\geqq 0$$

Therefore, $\boldsymbol{a\leqq\frac{3}{2}}$

(6) $x^2-(2a-1)x+1+a^2=0$

$$D=(2a-1)^2-4(1+a^2)$$
$$=4a^2-4a+1-4-4a^2$$
$$=-4a-3\geqq 0$$

Therefore, $\boldsymbol{a\leqq -\frac{3}{4}}$

(7) $x^2-2ax+(a+1)^2=0$

$$\frac{D}{4}=a^2-(a+1)^2$$
$$=a^2-a^2-2a-1$$
$$=-2a-1\geqq 0$$

Therefore, $\boldsymbol{a\leqq -\frac{1}{2}}$

(8)* $(a+1)x^2-2(a+1)x+(a-1)=0$
$(a\neq -1)$

$$\frac{D}{4}=(a+1)^2-(a+1)(a-1)$$
$$=a^2+2a+1-a^2+1$$
$$=2a+2\geqq 0$$

Therefore, $a\geqq -1$
Since $a\neq -1$,
the answer is $\boldsymbol{a>-1}$

J 126a

KUMON

Discriminant

Determine the value of k for which each equation has 2 different ***complex solutions*** $(D<0)$.

Ex.

$x^2-3x+3k=0$

[Sol] $D=9-12k<0$

To have 2 different ***complex solutions*** the discriminant must be less than zero, $D<0$.

Therefore, $k>\dfrac{3}{4}$

(1) $x^2-2x+3k=0$

[Sol] $\dfrac{D}{4}=1-3k<0$

Therefore, $k>\dfrac{1}{3}$

(2) $2x^2-3x+k-1=0$

$D=9-8(k-1)$
$=17-8k<0$

Therefore, $k>\dfrac{17}{8}$

(3) $2x^2-4x+k+1=0$

$\dfrac{D}{4}=4-2(k+1)$
$=2-2k<0$

Therefore, $k>1$

(4) $3x^2-5x+2-k=0$

$D=25-12(2-k)$
$=1+12k<0$

Therefore, $k<-\dfrac{1}{12}$

J126b

(5) $x^2-2(k-1)x+k^2=0$

$$\frac{D}{4}=(k-1)^2-k^2$$
$$=k^2-2k+1-k^2$$
$$=-2k+1<0$$

Therefore, $k>\frac{1}{2}$

(6) $x^2-(2k+1)x-1+k^2=0$

$$D=(2k+1)^2-4(-1+k^2)$$
$$=4k^2+4k+1+4-4k^2$$
$$=4k+5<0$$

Therefore, $k<-\frac{5}{4}$

(7) $x^2-4kx+(2k+1)^2=0$

$$\frac{D}{4}=4k^2-(2k+1)^2$$
$$=4k^2-4k^2-4k-1$$
$$=-4k-1<0$$

Therefore, $k>-\frac{1}{4}$

(8) $kx^2-2(k+1)x+(k-1)=0$

$(k\neq 0)$

$$\frac{D}{4}=(k+1)^2-k(k-1)$$
$$=k^2+2k+1-k^2+k$$
$$=3k+1<0$$

Therefore, $k<-\frac{1}{3}$

J I 27a

KUMON

Discriminant

Discriminate the solutions of the following quadratic equations.

Ex.

$x^2-5x+k=0$

[Sol] $D=25-4k$

Therefore,

When $k<\dfrac{25}{4}$,
there are 2 different real solutions. ☜ From $D=25-4k>0$.

When $k=\dfrac{25}{4}$,
there is a repeated real solution. ☜ From $D=25-4k=0$.

When $k>\dfrac{25}{4}$,
there are 2 different complex solutions. ☜ From $D=25-4k<0$.

(1) $x^2-3x+k=0$

$D=9-4k$

Therefore,

When $k<\dfrac{9}{4}$, there are 2 different real solutions.

When $k=\dfrac{9}{4}$, there is a repeated real solution.

When $k>\dfrac{9}{4}$, there are 2 different complex solutions.

J 127b

(2) $x^2+2x+3a=0$

$\dfrac{D}{4}=1-3a$

Therefore,

$$\begin{cases} \textbf{When } \boldsymbol{a}<\dfrac{1}{3}\textbf{, there are 2 different real solutions.} \\ \textbf{When } \boldsymbol{a}=\dfrac{1}{3}\textbf{, there is a repeated real solution.} \\ \textbf{When } \boldsymbol{a}>\dfrac{1}{3}\textbf{, there are 2 different complex solutions.} \end{cases}$$

(3) $2x^2-3x-k=0$

$D=9+8k$

Therefore,

$$\begin{cases} \textbf{When } \boldsymbol{k}>-\dfrac{9}{8}\textbf{, there are 2 different real solutions.} \\ \textbf{When } \boldsymbol{k}=-\dfrac{9}{8}\textbf{, there is a repeated real solution.} \\ \textbf{When } \boldsymbol{k}<-\dfrac{9}{8}\textbf{, there are 2 different complex solutions.} \end{cases}$$

(4) $2x^2-4x-3m=0$

$\dfrac{D}{4}=4+6m$

Therefore,

$$\begin{cases} \textbf{When } \boldsymbol{m}>-\dfrac{2}{3}\textbf{, there are 2 different real solutions.} \\ \textbf{When } \boldsymbol{m}=-\dfrac{2}{3}\textbf{, there is a repeated real solution.} \\ \textbf{When } \boldsymbol{m}<-\dfrac{2}{3}\textbf{, there are 2 different complex solutions.} \end{cases}$$

J 128a KUMON

Discriminant

Discriminate the solutions of the following quadratic equations.

Ex.

$2x^2+x-2a^2=0$

[Sol] $D=1+16a^2$

From $a^2 \geqslant 0$

$D=1+16a^2>0$

A squared real number cannot be negative. Thus, regardless of the value of a, $1+16a^2 \geqslant 1>0$.

Therefore, there are 2 different real solutions.

(1) $ax^2-3x-2a=0$

$D=9+8a^2$

From $a^2 \geqslant 0$

$D=9+8a^2>0$

Therefore, **there are 2 different real solutions.**

(2) $(a+5)x^2-3ax+(a-5)=0$

$D=9a^2-4(a+5)(a-5)=5a^2+100$

From $a^2 \geqslant 0$

$D=5a^2+100>0$

Therefore, **there are 2 different real solutions.**

(3)* $x^2-2ax+2a^2+1=0$

$\frac{D}{4}=a^2-2a^2-1=-(a^2+1)$

From $a^2 \geqslant 0$

$D=-(a^2+1)<0$

Therefore, **there are 2 different complex solutions.**

J128b

Ex.

$x^2-3ax+2a^2=0$

[Sol] $D=9a^2-8a^2=a^2$

Therefore,

When $a=0$, there is a repeated real solution. ☜ From $D=0$.

When $a\neq 0$, there are 2 different real solutions. ☜ From $D>0$.

(4) $x^2-4ax+3a^2=0$

$\frac{D}{4}=4a^2-3a^2=a^2$

Therefore,

$\begin{cases} \textbf{When } \boldsymbol{a=0}\textbf{, there is a repeated real solution.} \\ \textbf{When } \boldsymbol{a\neq 0}\textbf{, there are 2 different real solutions.} \end{cases}$

(5) $x^2-ax+a^2=0$

$D=a^2-4a^2=-3a^2$

Therefore,

$\begin{cases} \textbf{When } \boldsymbol{a=0}\textbf{, there is a repeated real solution.} \\ \textbf{When } \boldsymbol{a\neq 0}\textbf{, there are 2 different complex solutions.} \end{cases}$

(6) $x^2-2ax+a^2=0$

$\frac{D}{4}=a^2-a^2=0$

Therefore, **there is a repeated real solution for all real values of $\boldsymbol{a}$.**

J 129a

KUMON

Discriminant

Ex.

The equation $x^2+(m+1)x+(2m-1)=0$ has a repeated real solution.

(1) Find the value(s) of m.

[Sol] $D=(m+1)^2-4(2m-1)$

$=m^2-6m+5$

$=(m-5)(m-1)=0$

Thus, $m=5,\ 1$

(2) Find the repeated real solution(s).

[Sol] When $m=5$,

$x=-\dfrac{m+1}{2}=-3$

When $m=1$,

$x=-\dfrac{m+1}{2}=-1$

When $ax^2+bx+c=0$, $D=b^2-4ac=0$, therefore,

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=-\dfrac{b}{2a}$

1. The equation $3x^2+4mx+m=0$ has a repeated real solution. Find the value(s) of m, and the solution(s).

[Sol] $\dfrac{D}{4}=4m^2-3m$

$=m(4m-3)=0$

Thus, $m=0,\ \dfrac{3}{4}$

When $m=0$,

$x=-\dfrac{4m}{2\cdot3}=-\dfrac{0}{6}=\mathbf{0}$

When $m=\dfrac{3}{4}$,

$x=-\dfrac{4m}{2\cdot3}=-\dfrac{3}{6}=-\dfrac{1}{2}$

J 129b

2.* Given the equation $x^2+(2k-3)x+2k=0$ has a repeated negative real solution, find the solution.

[Sol] $D=(2k-3)^2-8k$

$=4k^2-12k+9-8k$

$=4k^2-20k+9=0$

Thus, $k=\dfrac{10\pm\sqrt{100-36}}{4}$

$=\dfrac{10\pm 8}{4}$

$k=\dfrac{9}{2},\ \dfrac{1}{2}$

When $k=\dfrac{9}{2}$,

$x=-\dfrac{2k-3}{2}=-\dfrac{6}{2}=-3$

When $k=\dfrac{1}{2}$,

$x=-\dfrac{2k-3}{2}=-\dfrac{1-3}{2}=1$

$x=1$ is positive and, thus, it is not the answer to this problem.

Thus, $\boldsymbol{x=-3}$

J I 30a KUMON

Discriminant

1. Discriminate the solutions of the following equations.

(1) $x^2+5x+3+a=0$

$D=25-4(3+a)$

$=13-4a$

Therefore,

$$\begin{cases} \textbf{When } a<\dfrac{13}{4}, & \textbf{there are 2 different real solutions.} \\ \textbf{When } a=\dfrac{13}{4}, & \textbf{there is a repeated real solution.} \\ \textbf{When } a>\dfrac{13}{4}, & \textbf{there are 2 different complex solutions.} \end{cases}$$

(2) $x^2+(a+b)x+ab=0$

$D=(a+b)^2-4ab$

$=a^2-2ab+b^2$

$=(a-b)^2$

Therefore,

$$\begin{cases} \textbf{When } a=b, & \textbf{there is a repeated real solution.} \\ \textbf{When } a\neq b, & \textbf{there are 2 different real solutions.} \end{cases}$$

J130b

2. The quadratic equation $(b-c)x^2+(c-a)x+(a-b)=0$ has a repeated real solution. Prove that $b=\dfrac{a+c}{2}$.

(Hint: We can rearrange $b=\dfrac{a+c}{2}$ as $a+c-2b=0$.)

[Sol] $D=(c-a)^2-4(b-c)(a-b)$
$=(c^2-2ac+a^2)-4(ab-b^2-ac+bc)$
$=c^2-2ac+a^2-4ab+4b^2+4ac-4bc$
$=a^2+2(c-2b)a+(c-2b)^2$
$=(a+c-2b)^2$

For a repeated real solution, $D=0$

$D=(a+c-2b)^2=0$

$a+c-2b=0$

Thus,

$$\boldsymbol{b=\frac{a+c}{2}}$$

J 131 a KUMON

Root-Coefficient Relationships

1. By first finding the roots of each quadratic equation, obtain the sum and product of these roots.

Ex.

$x^2-4x+3=0$

[Sol] $(x-3)(x-1)=0$

$x=3,\ 1$

$\alpha+\beta=3+1=4$ ☜ Use α and β for the two roots.

$\alpha\beta=3\times1=3$

(1) $x^2-2x-8=0$

$(x-4)(x+2)=0$

$x=4,\ -2$

$\alpha+\beta=4-2=\mathbf{2}$

$\alpha\beta=4\times(-2)=\mathbf{-8}$

(2) $x^2-7x+5=0$

$$x=\frac{7\pm\sqrt{49-20}}{2}=\frac{7\pm\sqrt{29}}{2}$$

$$\alpha+\beta=\frac{7+\sqrt{29}}{2}+\frac{7-\sqrt{29}}{2}=\mathbf{7}$$

$$\alpha\beta=\left(\frac{7+\sqrt{29}}{2}\right)\left(\frac{7-\sqrt{29}}{2}\right)=\mathbf{5}$$

(3) $3x^2+2x+5=0$

$$x=\frac{-1\pm\sqrt{1-15}}{3}=\frac{-1\pm\sqrt{14}\,i}{3}$$

$$\alpha+\beta=\frac{-1+\sqrt{14}i}{3}+\frac{-1-\sqrt{14}i}{3}=\mathbf{-\frac{2}{3}}$$

$$\alpha\beta=\left(\frac{-1+\sqrt{14}i}{3}\right)\left(\frac{-1-\sqrt{14}i}{3}\right)=\mathbf{\frac{5}{3}}$$

(4) $ax^2+bx+c=0$

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}=\boldsymbol{-\frac{b}{a}}$$

$$\alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)=\frac{b^2-b^2+4ac}{4a^2}=\boldsymbol{\frac{c}{a}}$$

J I 3 I b

From result of J I 3 I a (4), the formulas for the relationships between the roots and coefficients of quadratic equations are:

Root-Coefficient Relationships

Given $ax^2+bx+c=0$ $(a \neq 0)$, if the roots are α and β,

$$\alpha+\beta=-\frac{b}{a} \qquad \alpha\beta=\frac{c}{a}$$

2. If the roots of the quadratic equation are α and β, obtain their sum and product using the formulas above.

Ex.

$x^2-3x+9=0$

$\alpha+\beta=-\frac{-3}{1}=3$

$\alpha\beta=\frac{9}{1}=9$

(1) $3x^2-7x+6=0$

$\boldsymbol{\alpha+\beta=\frac{7}{3}}$

$\boldsymbol{\alpha\beta=2}$

(2) $3x^2+6x+7=0$

$\boldsymbol{\alpha+\beta=-2}$

$\boldsymbol{\alpha\beta=\frac{7}{3}}$

(3) $7x^2-6x+1=0$

$\boldsymbol{\alpha+\beta=\frac{6}{7}}$

$\boldsymbol{\alpha\beta=\frac{1}{7}}$

J I 32a KUMON

Root-Coefficient Relationships

Root-Coefficient Relationships

Given $ax^2+bx+c=0$ $(a\neq 0)$, if the roots are α and β,

$$\alpha+\beta=-\frac{b}{a} \qquad \alpha\beta=\frac{c}{a}$$

1. If the roots of the quadratic equation are α and β, obtain their sum and product using the formulas above.

(1) $3x^2+7x-6=0$

$\boldsymbol{\alpha+\beta=-\frac{7}{3}}$

$\boldsymbol{\alpha\beta=-2}$

(2) $x^2+3=0$

$\boldsymbol{\alpha+\beta=0}$

$\boldsymbol{\alpha\beta=3}$

(3) $5x^2-3x=0$

$\boldsymbol{\alpha+\beta=\frac{3}{5}}$

$\boldsymbol{\alpha\beta=0}$

(4) $x^2+ax+b=0$

$\boldsymbol{\alpha+\beta=-a}$

$\boldsymbol{\alpha\beta=b}$

(5) $x^2+\frac{b}{a}x+\frac{c}{a}=0$

$\boldsymbol{\alpha+\beta=-\frac{b}{a}}$

$\boldsymbol{\alpha\beta=\frac{c}{a}}$

(6) $x^2-\frac{5}{6}x+\frac{1}{6}=0$

$\boldsymbol{\alpha+\beta=\frac{5}{6}}$

$\boldsymbol{\alpha\beta=\frac{1}{6}}$

2. Obtain each quadratic equation using the 2 given roots.

Ex.

$\frac{1}{2}$, $\frac{1}{3}$

[Sol] If the roots are α and β,

$$\alpha+\beta=\frac{5}{6} \quad \alpha\beta=\frac{1}{6}$$

Therefore,

$$x^2-\frac{5}{6}x+\frac{1}{6}=0$$

$$6x^2-5x+1=0$$

See question (6) on side a — the same values are used.

Multiply both sides by 6.

(1) $-\frac{1}{4}$, $\frac{2}{3}$

If the roots are α and β,

$$\alpha+\beta=\frac{5}{12}$$

$$\alpha\beta=-\frac{1}{6}$$

Therefore,

$$x^2-\frac{5}{12}x-\frac{1}{6}=0$$

$$\boldsymbol{12x^2-5x-2=0}$$

(2) 2, $-\frac{2}{3}$

If the roots are α and β,

$$\alpha+\beta=\frac{4}{3}$$

$$\alpha\beta=-\frac{4}{3}$$

Therefore,

$$x^2-\frac{4}{3}x-\frac{4}{3}=0$$

$$\boldsymbol{3x^2-4x-4=0}$$

(3) 3, -5

If the roots are α and β,

$$\alpha+\beta=-2$$

$$\alpha\beta=-15$$

Therefore,

$$\boldsymbol{x^2+2x-15=0}$$

(4) -3, -8

If the roots are α and β,

$$\alpha+\beta=-11$$

$$\alpha\beta=24$$

Therefore,

$$\boldsymbol{x^2+11x+24=0}$$

J I33a KUMON

Root-Coefficient Relationships

Obtain each quadratic equation using the 2 given roots.

Ex.

$1+\sqrt{2},\ 1-\sqrt{2}$

[Sol] If the roots are α and β,

$$\alpha+\beta=2$$

$$\alpha\beta=-1$$

Therefore,

$$x^2-2x-1=0$$

(1) $3+\sqrt{5},\ 3-\sqrt{5}$

$\alpha+\beta=6$
$\alpha\beta=4$
$\boldsymbol{x^2-6x+4=0}$

(2) $-\sqrt{3},\ \sqrt{3}$

$\alpha+\beta=0$
$\alpha\beta=-3$
$\boldsymbol{x^2-3=0}$

(3) $2+\sqrt{3}\,i,\ 2-\sqrt{3}\,i$

$\alpha+\beta=4$
$\alpha\beta=7$
$\boldsymbol{x^2-4x+7=0}$

(4) $2+3i,\ 2-3i$

$\alpha+\beta=4$
$\alpha\beta=13$
$\boldsymbol{x^2-4x+13=0}$

(5) $1+i,\ 1-i$

$\alpha+\beta=2$
$\alpha\beta=2$
$\boldsymbol{x^2-2x+2=0}$

J133b

(6) $\dfrac{1+\sqrt{5}}{2}$, $\dfrac{1-\sqrt{5}}{2}$

$\alpha+\beta=1$
$\alpha\beta=-1$
$\boldsymbol{x^2-x-1=0}$

(8) $a+bi$, $a-bi$

$\alpha+\beta=2a$
$\alpha\beta=a^2+b^2$
$\boldsymbol{x^2-2ax+a^2+b^2=0}$

(7) a, b

$\alpha+\beta=a+b$
$\alpha\beta=ab$
$\boldsymbol{x^2-(a+b)x+ab=0}$

(9) $3a+2\sqrt{5}\,b$, $3a-2\sqrt{5}\,b$

$\alpha+\beta=6a$
$\alpha\beta=9a^2-20b^2$
$\boldsymbol{x^2-6ax+9a^2-20b^2=0}$

Note: A quadratic equation with solutions $\dfrac{1+\sqrt{5}}{2}$ and $\dfrac{1-\sqrt{5}}{2}$ can be written $\left(x-\dfrac{1+\sqrt{5}}{2}\right)\left(x-\dfrac{1-\sqrt{5}}{2}\right)=0.$

This can be expanded and simplified: however it is simpler to find the quadratic equation using the method shown.

J 134a KUMON

Root-Coefficient Relationships

1. If the roots of $x^2-4x+5=0$ are α and β, evaluate the following expressions.

Ex.

$\alpha^2+\beta^2$

[Sol] $\alpha+\beta=4$, $\alpha\beta=5$

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=4^2-2\times5=6$

(1) $(\alpha-\beta)^2$

[Sol] $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$
$=4^2-4\times5$
$=\mathbf{-4}$

(2) $\alpha^2\beta+\alpha\beta^2$

[Sol] $\alpha^2\beta+\alpha\beta^2=\alpha\beta(\alpha+\beta)$
$=5\times4$
$=\mathbf{20}$

(3) $\alpha^2+\alpha\beta+\beta^2=(\alpha+\beta)^2-\alpha\beta$
$=16-5$
$=\mathbf{11}$

2. If the roots of $3x^2+3x-1=0$ are α and β, evaluate the following expressions.

(1) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$

$\alpha+\beta=-1,\ \alpha\beta=-\dfrac{1}{3}$

$$\begin{aligned}\frac{1}{\alpha}+\frac{1}{\beta}&=\frac{1}{\alpha\beta}(\alpha+\beta)\\&=(-3)\times(-1)\\&=\mathbf{3}\end{aligned}$$

(2)
$$\begin{aligned}\frac{\beta}{\alpha}+\frac{\alpha}{\beta}&=\frac{1}{\alpha\beta}(\alpha^2+\beta^2)\\&=\frac{1}{\alpha\beta}[(\alpha+\beta)^2-2\alpha\beta]\\&=-3\times\left[(-1)^2-2\times\left(-\frac{1}{3}\right)\right]\\&=\mathbf{-5}\end{aligned}$$

(3)
$$\begin{aligned}\frac{1}{\alpha+1}+\frac{1}{\beta+1}&=\frac{(\beta+1)+(\alpha+1)}{(\alpha+1)(\beta+1)}\\&=\frac{(\alpha+\beta)+2}{\alpha\beta+(\alpha+\beta)+1}\\&=\frac{-1+2}{-\frac{1}{3}+(-1)+1}\\&=\mathbf{-3}\end{aligned}$$

J I 35a KUMON

Root-Coefficient Relationships

1. If the roots of $3x^2+3x-1=0$ are α and β, evaluate the following expressions.

(1) $\alpha^2+\beta^2$

$$\alpha+\beta=-1,\ \alpha\beta=-\frac{1}{3}$$

$$\begin{aligned}\alpha^2+\beta^2&=(\alpha+\beta)^2-2\alpha\beta\\&=(-1)^2-2\times\left(-\frac{1}{3}\right)\\&=\mathbf{\frac{5}{3}}\end{aligned}$$

(2) $$\begin{aligned}\alpha^3+\beta^3&=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)\\&=(-1)^3-3\times\left(-\frac{1}{3}\right)\times(-1)\\&=\mathbf{-2}\end{aligned}$$ *Hint*

(3) $$\begin{aligned}\alpha^4+\beta^4&=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2\\&=(\alpha^2+\beta^2)^2-2(\alpha\beta)^2\\&=\left(\frac{5}{3}\right)^2-2\times\left(-\frac{1}{3}\right)^2\\&=\mathbf{\frac{23}{9}}\end{aligned}$$ *Hint*

Hints

(2) $a^3+b^3=(a+b)^3-3ab(a+b)$
(3) $a^4+b^4=(a^2+b^2)^2-2a^2b^2$

J135b

2. If the roots of $x^2+5x-1=0$ are α and β, evaluate the following expressions.

(1) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$

$$\alpha+\beta=-5,\ \alpha\beta=-1$$

$$\begin{aligned}\frac{1}{\alpha^2}+\frac{1}{\beta^2}&=\frac{1}{\alpha^2\beta^2}(\alpha^2+\beta^2)\\&=\frac{1}{(\alpha\beta)^2}[(\alpha+\beta)^2-2\alpha\beta]\\&=1\times[(-5)^2-2\times(-1)]\\&=\mathbf{27}\end{aligned}$$

(2) $$\begin{aligned}\frac{1}{\alpha^3}+\frac{1}{\beta^3}&=\frac{1}{\alpha^3\beta^3}(\alpha^3+\beta^3)\\&=\frac{1}{(\alpha\beta)^3}[(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)]\\&=(-1)\times[(-5)^3-3\times(-1)\times(-5)]\\&=\mathbf{140}\end{aligned}$$

$$\begin{aligned}\alpha^2+\beta^2&=(\alpha+\beta)^2-2\alpha\beta\\&=(-5)^2-2\times(-1)\\&=27\end{aligned}$$

(3) $$\begin{aligned}\frac{1}{\alpha^4}+\frac{1}{\beta^4}&=\frac{1}{\alpha^4\beta^4}(\alpha^4+\beta^4)\\&=\frac{1}{(\alpha\beta)^4}[(\alpha^2+\beta^2)^2-2\alpha^2\beta^2]\\&=1\times[27^2-2\times(-1)^2]\\&=\mathbf{727}\end{aligned}$$

J136a KUMON

Root-Coefficient Relationships

Ex.

If the roots of $x^2-4x+7=0$ are α and β, create a quadratic equation that has the roots α^2 and β^2. (Note that the coefficient of x^2 is 1.)

[Sol] $\alpha+\beta=4$

$\alpha\beta=7$

From the root-coefficient relationships of the given quadratic equation.

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$=16-2\cdot7=2 \cdots①$

$\alpha^2\beta^2=(\alpha\beta)^2=49 \cdots②$

From the relationships between the sum and product of α^2 and β^2.

From ① and ②,

$x^2-2x+49=0$

1. If the roots of $x^2-4x+7=0$ are α and β, create a quadratic equation that has the roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$. (Note that the coefficient of x^2 is 1.)

[Sol] $\alpha+\beta=4$

$\alpha\beta=7$

$\dfrac{1}{\alpha}+\dfrac{1}{\beta}=\dfrac{1}{\alpha\beta}(\alpha+\beta)$

$=\dfrac{1}{7}\cdot4=\dfrac{4}{7} \cdots①$

$\dfrac{1}{\alpha}\cdot\dfrac{1}{\beta}=\dfrac{1}{\alpha\beta}=\dfrac{1}{7} \cdots②$

From ① and ②,

$$\boldsymbol{x^2-\frac{4}{7}x+\frac{1}{7}=0}$$

J136b

2. If the roots of $x^2-x+1=0$ are α and β, create a quadratic equation that has the roots $\dfrac{1}{\alpha+1}$ and $\dfrac{1}{\beta+1}$. (Note that the coefficient of x^2 is 1.)

[Sol] $\alpha+\beta=1$

$\alpha\beta=1$

$$\frac{1}{\alpha+1}+\frac{1}{\beta+1}=\frac{(\beta+1)+(\alpha+1)}{(\alpha+1)(\beta+1)}$$

$$=\frac{(\alpha+\beta)+2}{\alpha\beta+(\alpha+\beta)+1}$$

$$=1 \quad \cdots①$$

$$\left(\frac{1}{\alpha+1}\right)\left(\frac{1}{\beta+1}\right)=\frac{1}{\alpha\beta+(\alpha+\beta)+1}$$

$$=\frac{1}{3} \quad \cdots②$$

From ① and ②,

$$\boldsymbol{x^2-x+\frac{1}{3}=0}$$

J 137a KUMON

Root-Coefficient Relationships

1. Given that $x^2-3x+1=0$ has 2 different roots, create a quadratic equation with roots 1 greater than the corresponding roots of the original equation.

[Sol] If the roots of $x^2-3x+1=0$ are α and β, *Hint*

$$\alpha+\beta=3$$

$$\alpha\beta=1$$

Find the sum and product of the roots of the new quadratic equation.

Therefore,

$$(\alpha+1)+(\beta+1)=(\alpha+\beta)+2=5 \quad \cdots①$$

$$(\alpha+1)(\beta+1)=\alpha\beta+(\alpha+\beta)+1=5 \quad \cdots②$$

From ① and ②,

$$\boldsymbol{x^2-5x+5=0}$$

Hint

If the roots of $x^2-3x+1=0$ are α and β, the roots of the new equation will be $\alpha+1$ and $\beta+1$.

2. Given that $x^2-2x-5=0$ has 2 different roots, and $x^2+ax+b=0$ has 2 roots, each 2 less than the corresponding root of the equation, find the values of a and b.

[Sol] If the roots of $x^2-2x-5=0$ are α and β, *Hint*

$$\alpha+\beta=2$$

$$\alpha\beta=-5$$

Find the sum and product of the roots of the second quadratic equation.

Therefore,

$$(\alpha-2)+(\beta-2)=(\alpha+\beta)-4=-2 \quad \cdots①$$

$$(\alpha-2)(\beta-2)=\alpha\beta-2(\alpha+\beta)+4=-5 \quad \cdots②$$

From ① and ②,

$$x^2+2x-5=0$$

Since the second equation is $x^2+ax+b=0$,

$$\boldsymbol{a=2} \text{ and } \boldsymbol{b=-5}$$

Hint

If the roots of $x^2-2x-5=0$ are α and β, the roots of the second quadratic equation will be $\alpha-2$ and $\beta-2$.

J 138a KUMON

Root-Coefficient Relationships

Ex.

Given that $x^2-6x+k=0$ has roots α and α^2, find the value of α, then find the value of k.

[Sol] $\begin{cases} \alpha+\alpha^2=6 & \cdots① \\ \alpha\cdot\alpha^2=\alpha^3=k & \cdots② \end{cases}$

From the Root-Coefficient Relationships of the given quadratic equation.

From ①,

$\alpha^2+\alpha-6=0$

$(\alpha+3)(\alpha-2)=0$

Therefore, $\alpha=-3,\ 2$

Substituting into ②,

When $\alpha=-3$, $k=-27$

When $\alpha=2$, $k=8$

1. Given that $x^2-12x+k=0$ has roots α and α^2, find the value of α, then find the value of k.

[Sol] $\begin{cases} \alpha+\alpha^2=12 & \cdots① \\ \alpha\cdot\alpha^2=\alpha^3=k & \cdots② \end{cases}$

From ①,

$\alpha^2+\alpha-12=0$

$(\alpha+4)(\alpha-3)=0$

$\alpha=-4,\ 3$

Substituting into ②,

When $\alpha=-4$, $k=-64$

When $\alpha=3$, $k=27$

2. Given that $x^2-12x+k=0$ has roots α and 2α, find the value of α, then find the value of k.

[Sol] $\begin{cases} \alpha+2\alpha=12 & \cdots① \\ \alpha\cdot 2\alpha=2\alpha^2=k & \cdots② \end{cases}$

From ①,

$3\alpha=12$

$\boldsymbol{\alpha=4}$

Substituting into ②,

$\boldsymbol{k=32}$

3. Given that $x^2+kx+12=0$ has roots α and 2α, find the value of α, then find the value of k.

[Sol] $\begin{cases} \alpha+2\alpha=-k & \cdots① \\ \alpha\cdot 2\alpha=12 & \cdots② \end{cases}$

From ②,

$\alpha^2=6$

$\alpha=\pm\sqrt{6}$

Substituting into ①,

When $\boldsymbol{\alpha=\sqrt{6}}$, $\boldsymbol{k=-3\sqrt{6}}$

When $\boldsymbol{\alpha=-\sqrt{6}}$, $\boldsymbol{k=3\sqrt{6}}$

J 139a KUMON

Root-Coefficient Relationships

Ex.

Determine the range of k for which $x^2-6x+3k=0$ has 2 different positive real solutions.

[Sol] If $x^2-6x+3k=0$ have roots α and β.

$\alpha+\beta=6$, $\alpha\beta=3k$ ☜ From the Root-Coefficient Relationships.

From $\alpha\beta=3k>0$, ☜ If α and β are both positive, then $\alpha\beta$ must be positive.

$k>0 \quad \cdots ①$

And, from $\dfrac{D}{4}=9-3k>0$, ☜ In order to have 2 different real number solutions, $D>0$.

$k<3 \quad \cdots ②$

From ① and ②,

$0<k<3$

1. Determine the range of k for which $3x^2+8x+2k=0$ has 2 different negative real solutions.

[Sol] If $3x^2+8x+2k=0$ have roots α and β.

$\alpha+\beta=-\dfrac{8}{3}$, $\alpha\beta=\dfrac{2}{3}k$

From $\alpha\beta=\dfrac{2}{3}k>0$, ☜ If α and β are both negative, then $\alpha\beta$ must be positive.

$k>0 \cdots ①$

And, from $\dfrac{D}{4}=16-6k>0$,

$k<\dfrac{8}{3}\cdots ②$

From ① and ②,

$\boldsymbol{0<k<\dfrac{8}{3}}$

J139b

2. Given that $x^2-2x-m+1=0$ has 2 positive real number roots, α and β, determine the range of values of m.

[Sol] $\alpha+\beta=2$, $\alpha\beta=-m+1$

From $\alpha\beta=-m+1>0$,

$m<1 \quad \cdots ①$

And, from $\dfrac{D}{4}=1-(-m+1)>0$,

$m>0 \quad \cdots ②$

From ① and ②,

$\mathbf{0<m<1}$

Consider this!

i) Given that $x^2-6x+3k=0$ has 2 real number roots, α and β, with opposite signs (one root is positive and the other root is negative), determine the range of values of k.

[Sol] $\alpha+\beta=6$, $\alpha\beta=3k$

From $\alpha\beta=3k<\boxed{0}$,

A positive number times a negative number is negative.

$k<\boxed{0} \cdots ①$

And, from $\dfrac{D}{4}=9-3k>0$,

$k<3 \quad \cdots ②$

From ① and ②,

$k<\boxed{0}$

ii) In order for $ax^2+bx+c=0$ to have 2 different real number roots, α and β, with opposite signs,

$$\begin{cases} \alpha\beta=\dfrac{c}{a}<0 \\ D=b^2-4ac>0 \end{cases}$$

- If $\dfrac{c}{a}<0$, then $ac<0$, so the discriminant is always positive.
- Therefore we only need the condition $\alpha\beta<0$.

Answers: 0, 0, 0

J 140a KUMON

Root-Coefficient Relationships

1. Find the quadratic equation with the two given roots.

(1) $\dfrac{2+\sqrt{5}}{3}$, $\dfrac{2-\sqrt{5}}{3}$

$\alpha+\beta=\dfrac{4}{3}$

$\alpha\beta=-\dfrac{1}{9}$

Therefore,

$x^2-\dfrac{4}{3}x-\dfrac{1}{9}=0$

$\mathbf{9x^2-12x-1=0}$ Multiply by 9 to eliminate the denominator.

(2) $\dfrac{-1+\sqrt{3}\,i}{2}$, $\dfrac{-1-\sqrt{3}\,i}{2}$

$\alpha+\beta=-1$

$\alpha\beta=1$

Therefore,

$\mathbf{x^2+x+1=0}$

2. Given that $x^2+5x-1=0$ has 2 roots α and β, evaluate the following expressions.

(1) $\alpha^2-\alpha\beta+\beta^2$

$\alpha+\beta=-5$

$\alpha\beta=-1$

$$\begin{aligned}\alpha^2-\alpha\beta+\beta^2&=\alpha^2+\beta^2-\alpha\beta\\&=(\alpha+\beta)^2-3\alpha\beta\\&=25-3\times(-1)=\mathbf{28}\end{aligned}$$

(2) $$\begin{aligned}(\alpha-2\beta)(2\alpha-\beta)&=2\alpha^2-5\alpha\beta+2\beta^2\\&=2(\alpha+\beta)^2-9\alpha\beta\\&=2\times25-9\times(-1)=\mathbf{59}\end{aligned}$$

J140b

3. Given that $x^2+2mx+m^2-2m+3=0$ has 2 roots α and β (with $\alpha<\beta$),

(1) Find $\alpha+\beta$ and $\alpha\beta$ in terms of m.

$\boldsymbol{\alpha+\beta=-2m}$
$\boldsymbol{\alpha\beta=m^2-2m+3}$

(2) Find $(\alpha-\beta)^2$ in terms of m.

$$\begin{aligned}\boldsymbol{(\alpha-\beta)^2} &= (\alpha+\beta)^2-4\alpha\beta\\ &= 4m^2-4(m^2-2m+3)\\ &\boldsymbol{=8m-12}\end{aligned}$$

(3) If $\beta-\alpha=2$, find m.

From question (2),
$(\alpha-\beta)^2=8m-12$ ⋯①

We can also write the equation:
$(\alpha-\beta)^2=(\beta-\alpha)^2$ ⋯②

From ① and ②,
$8m-12=(\beta-\alpha)^2$
$8m-12=2^2$
$\boldsymbol{m=2}$

J 141 a KUMON

Simultaneous Equations

Solve the following simultaneous equations.

Ex.

$$\begin{cases} y = 2x - 1 & \cdots\cdots ① \\ x^2 + y^2 = 29 & \cdots\cdots ② \end{cases}$$

[Sol] Substituting ① into ②,

$$x^2 + (2x-1)^2 = 29$$
$$5x^2 - 4x - 28 = 0$$
$$(5x-14)(x+2) = 0$$

Therefore,

$$x = \frac{14}{5},\ -2$$

Substituting $x = \frac{14}{5}$ into ①,

$$y = \frac{23}{5}$$

Substituting $x = -2$ into ①,

$$y = -5$$

Ans. $\begin{cases} x = \frac{14}{5} \\ y = \frac{23}{5} \end{cases}$ $\begin{cases} x = -2 \\ y = -5 \end{cases}$

(1) $\begin{cases} y = 2x - 1 & \cdots\cdots ① \\ x^2 + y^2 = 13 & \cdots\cdots ② \end{cases}$

Note: Solving a group of simultaneous equations is often called solving a "system of equations".

Substituting ① into ②,

$$x^2 + (2x-1)^2 = 13$$
$$5x^2 - 4x - 12 = 0$$
$$(5x+6)(x-2) = 0$$

Therefore,

$$x = -\frac{6}{5},\ 2$$

Substituting $x = -\frac{6}{5}$ into ①,

$$y = -\frac{17}{5}$$

Substituting $x = 2$ into ①,

$$y = 3$$

Ans. $\begin{cases} \boldsymbol{x = -\frac{6}{5}} \\ \boldsymbol{y = -\frac{17}{5}} \end{cases}$ $\begin{cases} \boldsymbol{x = 2} \\ \boldsymbol{y = 3} \end{cases}$

(2) $\begin{cases} y = x+3 & \cdots\cdots① \\ x^2+y^2=5 & \cdots\cdots② \end{cases}$

Substituting ① into ②,

$x^2+(x+3)^2=5$

$2x^2+6x+4=0$

$x^2+3x+2=0$

$(x+2)(x+1)=0$

Therefore,

$x=-2,\ -1$

Substituting $x=-2$ into ①,

$y=1$

Substituting $x=-1$ into ①,

$y=2$

Ans. $\begin{cases} \boldsymbol{x=-2} \\ \boldsymbol{y=1} \end{cases}$ $\begin{cases} \boldsymbol{x=-1} \\ \boldsymbol{y=2} \end{cases}$

(3) $\begin{cases} x = y+1 & \cdots\cdots① \\ x^2+y^2=5 & \cdots\cdots② \end{cases}$

Substituting ① into ②,

$(y+1)^2+y^2=5$

$2y^2+2y-4=0$

$y^2+y-2=0$

$(y+2)(y-1)=0$

Therefore,

$y=-2,\ 1$

Substituting $y=-2$ into ①,

$x=-1$

Substituting $y=1$ into ①,

$x=2$

Ans. $\begin{cases} \boldsymbol{x=-1} \\ \boldsymbol{y=-2} \end{cases}$ $\begin{cases} \boldsymbol{x=2} \\ \boldsymbol{y=1} \end{cases}$

J 142a KUMON

Simultaneous Equations

Solve the following simultaneous equations.

(1) $\begin{cases} y=2x+1 & \cdots\cdots ① \\ x^2-y^2=-21 & \cdots\cdots ② \end{cases}$

Substituting ① into ②,

$$x^2-(2x+1)^2=-21$$
$$3x^2+4x-20=0$$
$$(3x+10)(x-2)=0$$

Therefore,

$$x=-\frac{10}{3},\ 2$$

Substituting $x=-\frac{10}{3}$ into ①,

$$y=-\frac{17}{3}$$

Substituting $x=2$ into ①,

$$y=5$$

Ans. $\begin{cases} \boldsymbol{x=-\frac{10}{3}} \\ \boldsymbol{y=-\frac{17}{3}} \end{cases}$ $\begin{cases} \boldsymbol{x=2} \\ \boldsymbol{y=5} \end{cases}$

(2) $\begin{cases} y=2x-1 & \cdots\cdots ① \\ x^2-2y^2=-1 & \cdots\cdots ② \end{cases}$

Substituting ① into ②,

$$x^2-2(2x-1)^2=-1$$
$$7x^2-8x+1=0$$
$$(7x-1)(x-1)=0$$

Therefore,

$$x=\frac{1}{7},\ 1$$

Substituting $x=\frac{1}{7}$ into ①,

$$y=-\frac{5}{7}$$

Substituting $x=1$ into ①,

$$y=1$$

Ans. $\begin{cases} \boldsymbol{x=\frac{1}{7}} \\ \boldsymbol{y=-\frac{5}{7}} \end{cases}$ $\begin{cases} \boldsymbol{x=1} \\ \boldsymbol{y=1} \end{cases}$

(3) $\begin{cases} y = x - 3 & \cdots\cdots ① \\ x^2 - y^2 = 15 & \cdots\cdots ② \end{cases}$

Substituting ① into ②,

$x^2 - (x-3)^2 = 15$

$6x = 24$

$x = 4$

Substituting $x = 4$ into ①,

$y = 1$

There is only one solution.

Ans. $\begin{cases} \boldsymbol{x = 4} \\ \boldsymbol{y = 1} \end{cases}$

(4) $\begin{cases} 2x - y = 3 & \cdots\cdots ① \\ x^2 - y^2 = 3 & \cdots\cdots ② \end{cases}$

After rearranging ①,

$y = 2x - 3 \quad \cdots\cdots ③$

Substituting ③ into ②,

$x^2 - (2x-3)^2 = 3$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x = 2$

Substituting $x = 2$ into ③,

$y = 1$

Ans. $\begin{cases} \boldsymbol{x = 2} \\ \boldsymbol{y = 1} \end{cases}$

J 143a KUMON

Simultaneous Equations

Solve the following simultaneous equations.

(1) $\begin{cases} 3x = 2y & \cdots\cdots ① \\ \dfrac{x^2}{4} + \dfrac{y^2}{9} = 8 & \cdots\cdots ② \end{cases}$

From ①,

$$x = \frac{2}{3}y \qquad \cdots\cdots ③$$

Substituting, ③ into ②,

$$\frac{\left(\frac{2}{3}y\right)^2}{4} + \frac{y^2}{9} = 8$$

$$\frac{1}{4} \times \frac{4}{9}y^2 + \frac{y^2}{9} = 8$$

$$y^2 = 36$$

$$y = \pm 6$$

Substituting $y = 6$ into ③,

$$x = 4$$

Substituting $y = -6$ into ③,

$$x = -4$$

Ans. $\begin{cases} x = \pm \boxed{4} \\ y = \pm \boxed{6} \end{cases}$

Solutions with the same numbers but different signs can be written as follows:

$\begin{cases} x = 1 \\ y = 2 \end{cases}$ and $\begin{cases} x = -1 \\ y = -2 \end{cases}$ can be written as $\begin{cases} x = \pm 1 \\ y = \pm 2 \end{cases}$

$\begin{cases} x = 1 \\ y = -2 \end{cases}$ and $\begin{cases} x = -1 \\ y = 2 \end{cases}$ can be written as $\begin{cases} x = \pm 1 \\ y = \mp 2 \end{cases}$

Keep the positive and negative signs in the correct order.

(2) $\begin{cases} y = x+1 & \cdots\cdots① \\ x^2+y^2=3 & \cdots\cdots② \end{cases}$

The solutions are irrational numbers.

Substituting ① into ②,

$$x^2+(x+1)^2=3$$
$$2x^2+2x-2=0$$
$$x^2+x-1=0$$

Therefore,

$$x=\frac{-1\pm\sqrt{5}}{2}$$

Substituting $x=\dfrac{-1+\sqrt{5}}{2}$ into ①,

$$y=\frac{1+\sqrt{5}}{2}$$

Substituting $x=\dfrac{-1-\sqrt{5}}{2}$ into ①,

$$y=\frac{1-\sqrt{5}}{2}$$

Ans. $\begin{cases} \boldsymbol{x=\dfrac{-1\pm\sqrt{5}}{2}} \\ \boldsymbol{y=\dfrac{1\pm\sqrt{5}}{2}} \end{cases}$

(3) $\begin{cases} 4x-2y=2 & \cdots\cdots① \\ x^2-y^2+3=0 & \cdots\cdots② \end{cases}$

The solutions are irrational numbers.

From ①,

$$y=2x-1 \quad \cdots\cdots③$$

Substituting ③ into ②,

$$x^2-(2x-1)^2+3=0$$
$$3x^2-4x-2=0$$

Therefore,

$$x=\frac{2\pm\sqrt{10}}{3}$$

Substituting $x=\dfrac{2+\sqrt{10}}{3}$ into ③,

$$y=\frac{1+2\sqrt{10}}{3}$$

Substituting $x=\dfrac{2-\sqrt{10}}{3}$ into ③,

$$y=\frac{1-2\sqrt{10}}{3}$$

Ans. $\begin{cases} \boldsymbol{x=\dfrac{2\pm\sqrt{10}}{3}} \\ \boldsymbol{y=\dfrac{1\pm2\sqrt{10}}{3}} \end{cases}$

J 144a KUMON

Simultaneous Equations

Solve the following simultaneous equations.

Ex.

$$\begin{cases} xy+x-y=1 & \cdots\cdots ① \\ xy-3x+y=5 & \cdots\cdots ② \end{cases}$$

[Sol] From ①−②,

$4x-2y=-4$

Therefore, $y=2x+2$ ……③

Substituting ③ into ①,

$x(2x+2)+x-(2x+2)=1$

$2x^2+x-3=0$

$(2x+3)(x-1)=0$

Therefore,

$x=-\dfrac{3}{2}$, 1

Substituting $x=-\dfrac{3}{2}$ into ③,

$y=-1$

Substituting $x=1$ into ③,

$y=4$

Ans. $\begin{cases} x=-\dfrac{3}{2} \\ y=-1 \end{cases}$ $\begin{cases} x=1 \\ y=4 \end{cases}$

(1) $\begin{cases} xy+3x-3y+3=0 & \cdots\cdots ① \\ xy-x-y+1=0 & \cdots\cdots ② \end{cases}$

From ①−②,

$4x-2y+2=0$

Therefore, $y=2x+1$……③

Substituting ③ into ①,

$x(2x+1)+3x-3(2x+1)+3=0$

$2x^2+x+3x-6x-3+3=0$

$2x^2-2x=0$

$x(x-1)=0$

Therefore,

$x=0$, 1

Substituting $x=0$ into ③,

$y=1$

Substituting $x=1$ into ③,

$y=3$

Ans. $\begin{cases} \boldsymbol{x=0} \\ \boldsymbol{y=1} \end{cases}$ $\begin{cases} \boldsymbol{x=1} \\ \boldsymbol{y=3} \end{cases}$

(2) $\begin{cases} xy+x+y=5 & \cdots\cdots① \\ 2xy+x+3y=9 & \cdots\cdots② \end{cases}$

From ①×2,

$2xy+2x+2y=10 \cdots\cdots③$

③−② : $x-y=1$

Therefore, $y=x-1 \quad \cdots\cdots④$

Substituting ④ into ①,

$x(x-1)+x+(x-1)=5$

$x^2+x-6=0$

$(x+3)(x-2)=0$

Therefore,

$x=-3,\ 2$

Substituting $x=-3$ into ④,

$y=-4$

Substituting $x=2$ into ④,

$y=1$

Ans. $\begin{cases} \boldsymbol{x=-3} \\ \boldsymbol{y=-4} \end{cases}$ $\begin{cases} \boldsymbol{x=2} \\ \boldsymbol{y=1} \end{cases}$

(3) $\begin{cases} xy+2x+2y+1=0 & \cdots\cdots① \\ 3xy+3x+5y+3=0 & \cdots\cdots② \end{cases}$

From ①×3,

$3xy+6x+6y+3=0 \cdots③$

③−② : $3x+y=0$

Therefore, $y=-3x \quad \cdots④$

Substituting ④ into ①,

$x(-3x)+2x+2(-3x)+1=0$

$-3x^2-4x+1=0$

$3x^2+4x-1=0$

Therefore,

$x=\dfrac{-2\pm\sqrt{7}}{3}$

Substituting $x=\dfrac{-2+\sqrt{7}}{3}$ into ④,

$y=2-\sqrt{7}$

Substituting $x=\dfrac{-2-\sqrt{7}}{3}$ into ④,

$y=2+\sqrt{7}$

Ans. $\begin{cases} \boldsymbol{x=\dfrac{-2\pm\sqrt{7}}{3}} \\ \boldsymbol{y=2\mp\sqrt{7}} \end{cases}$

J 145a KUMON

Simultaneous Equations

Solve the following equations.

Ex.

$$\begin{cases} x^2+y^2-2x+2y=7 & \cdots\cdots① \\ x^2+y^2+4x-4y=1 & \cdots\cdots② \end{cases}$$

[Sol] From ①−②,

$$-6x+6y=6$$

Therefore, $x=y-1$ ······③

Substituting ③ into ①,

$$(y-1)^2+y^2-2(y-1)+2y=7$$

$$2y^2-2y-4=0$$

$$(y-2)(y+1)=0$$

Therefore, $y=2,\ -1$

Substituting $y=2$ into ③,

$x=1$

Substituting $y=-1$ into ③,

$x=-2$

Ans. $\begin{cases} x=1 \\ y=2 \end{cases}$ $\begin{cases} x=-2 \\ y=-1 \end{cases}$

(1) $\begin{cases} x^2+y^2-4x-4y-10=0 & \cdots\cdots① \\ x^2+y^2+4x+4y-10=0 & \cdots\cdots② \end{cases}$

From ①−②,

$$-8x-8y=0$$

Therefore, $x=-y$ ······③

Substituting ③ into ①,

$$y^2+y^2+4y-4y-10=0$$

$$y^2=5$$

Therefore, $y=\pm\sqrt{5}$

Substituting $y=\sqrt{5}$ into ③,

$x=-\sqrt{5}$

Substituting $y=-\sqrt{5}$ into ③,

$x=\sqrt{5}$

Ans. $\begin{cases} \boldsymbol{x=\pm\sqrt{5}} \\ \boldsymbol{y=\mp\sqrt{5}} \end{cases}$

J145b

(2) $\begin{cases} x^2+y^2+x-y-6=0 & \cdots\cdots ① \\ 2x^2+2y^2+3x+y-12=0 & \cdots\cdots ② \end{cases}$

From ①×2,

$2x^2+2y^2+2x-2y-12=0 \cdots ③$

③−②: $-x-3y=0$

$x=-3y \quad \cdots\cdots ④$

Substituting ④ into ①,

$9y^2+y^2-3y-y-6=0$

$10y^2-4y-6=0$

$(5y+3)(y-1)=0$

Therefore, $y=-\dfrac{3}{5}$, 1

Substituting $y=-\dfrac{3}{5}$ into ④,

$x=\dfrac{9}{5}$

Substituting $y=1$ into ④,

$x=-3$

Ans. $\begin{cases} \boldsymbol{x=\dfrac{9}{5}} \\ \boldsymbol{y=-\dfrac{3}{5}} \end{cases}$ $\begin{cases} \boldsymbol{x=-3} \\ \boldsymbol{y=1} \end{cases}$

(3) $\begin{cases} 3x^2+2y^2-x+4y-8=0 & \cdots\cdots ① \\ 6x^2+4y^2+4x+11y-25=0 & \cdots\cdots ② \end{cases}$

From ①×2,

$6x^2+4y^2-2x+8y-16=0 \cdots\cdots ③$

③−②: $-6x-3y+9=0$

$y=-2x+3 \quad \cdots\cdots ④$

Substituting ④ into ①,

$3x^2+2(-2x+3)^2-x+4(-2x+3)-8=0$

$11x^2-33x+22=0$

$(x-2)(x-1)=0$

Therefore, $x=2$, 1

Substituting $x=2$ into ④,

$y=-1$

Substituting $x=1$ into ④,

$y=1$

Ans. $\begin{cases} \boldsymbol{x=2} \\ \boldsymbol{y=-1} \end{cases}$ $\begin{cases} \boldsymbol{x=1} \\ \boldsymbol{y=1} \end{cases}$

J 146a KUMON

Simultaneous Equations

Solve the following equations.

Ex.

$$\begin{cases} (x-y)(x+2y)=0 & \cdots\cdots① \\ x^2-xy+2y^2=16 & \cdots\cdots② \end{cases}$$

[Sol] From ①, $x=y$ or $x=-2y$

$$\begin{cases} x=y & \cdots③ \\ x^2-xy+2y^2=16 & \cdots② \end{cases}$$

Substituting ③ into ②,

$$y^2-y^2+2y^2=16$$
$$y^2=8$$
$$y=\pm 2\sqrt{2}$$

From ③, $x=\pm 2\sqrt{2}$

$$\begin{cases} x=-2y & \cdots③' \\ x^2-xy+2y^2=16 & \cdots② \end{cases}$$

Substituting ③′ into ②,

$$4y^2+2y^2+2y^2=16$$
$$y^2=2$$
$$y=\pm\sqrt{2}$$

From ③′, $x=\mp 2\sqrt{2}$

Ans. $\begin{cases} x=\pm 2\sqrt{2} \\ y=\pm 2\sqrt{2} \end{cases}$ $\begin{cases} x=\pm 2\sqrt{2} \\ y=\mp\sqrt{2} \end{cases}$

(1) $\begin{cases} (x-3y)(x-2y)=0 & \cdots\cdots① \\ x^2-2xy+6y^2=6 & \cdots\cdots② \end{cases}$

From ①, $x=3y$ or $x=2y$

$$\begin{cases} x=3y & \cdots③ \\ x^2-2xy+6y^2=6 & \cdots② \end{cases}$$

Substituting ③ into ②,

$$9y^2-6y^2+6y^2=6$$
$$y=\pm\frac{\sqrt{6}}{3}$$

From ③, $x=\pm\sqrt{6}$

$$\begin{cases} x=2y & \cdots③' \\ x^2-2xy+6y^2=6 & \cdots② \end{cases}$$

Substituting ③′ into ②,

$$4y^2-4y^2+6y^2=6$$
$$y=\pm 1$$

From ③′, $x=\pm 2$

Ans. $\begin{cases} \boldsymbol{x=\pm\sqrt{6}} \\ \boldsymbol{y=\pm\frac{\sqrt{6}}{3}} \end{cases}$ $\begin{cases} \boldsymbol{x=\pm 2} \\ \boldsymbol{y=\pm 1} \end{cases}$

Ex.

$$\begin{cases} x^2-4xy+3y^2=0 & \cdots\cdots① \\ x^2-5xy+6y^2=2 & \cdots\cdots② \end{cases}$$

[Sol] From ①, $(x-y)(x-3y)=0$ therefore $x=y$ or $x=3y$

$$\begin{cases} x=y & \cdots③ \\ x^2-5xy+6y^2=2 & \cdots② \end{cases}$$

Substituting ③ into ②,

$$y^2-5y^2+6y^2=2$$

$$y=\pm 1$$

From ③, $x=\pm 1$

$$\begin{cases} x=3y & \cdots③' \\ x^2-5xy+6y^2=2 & \cdots② \end{cases}$$

Substituting ③′ into ②,

$$9y^2-15y^2+6y^2=2$$

$$0\cdot y^2=2$$

There is no value of y that would satisfy this. Therefore, there is <u>no solution</u>.

Ans. $\begin{cases} x=\pm 1 \\ y=\pm 1 \end{cases}$

(2) $\begin{cases} x^2+3xy+2y^2=0 & \cdots\cdots① \\ x^2-2xy-3y^2=5 & \cdots\cdots② \end{cases}$

From ①, $(x+2y)(x+y)=0$ therefore $x=-2y$ or $x=-y$

$$\begin{cases} x=-2y & \cdots③ \\ x^2-2xy-3y^2=5 & \cdots② \end{cases}$$

Substituting ③ into ②,

$$4y^2+4y^2-3y^2=5$$

$$y=\pm 1$$

From ③, $x=\mp 2$

$$\begin{cases} x=-y & \cdots③' \\ x^2-2xy-3y^2=5 & \cdots② \end{cases}$$

Substituting ③′ into ②,

$$y^2+2y^2-3y^2=5$$

$$0\cdot y^2=5$$

There is no value of y that would satisfy this. Therefore, there is <u>no solution</u>.

Ans. $\begin{cases} \boldsymbol{x=\pm 2} \\ \boldsymbol{y=\mp 1} \end{cases}$

J 147a KUMON

Simultaneous Equations

Solve the following equations.

Ex.

$$\begin{cases} x+y=9 & \cdots\cdots ① \\ xy=7 & \cdots\cdots ② \end{cases}$$

[Sol] x and y are the roots of

$$t^2-9t+7=0$$

Therefore, $t=\dfrac{9\pm\sqrt{53}}{2}$ ☜

When a quadratic equation has roots α and β, we can write the equation as
$t^2-(\alpha+\beta)t+\alpha\beta=0$
In this case, $\alpha=x$ and $\beta=y$.

Ans. $\begin{cases} x=\dfrac{9+\sqrt{53}}{2} \\ y=\dfrac{9-\sqrt{53}}{2} \end{cases}$ $\begin{cases} x=\dfrac{9-\sqrt{53}}{2} \\ y=\dfrac{9+\sqrt{53}}{2} \end{cases}$

(1) $\begin{cases} x+y=4 \\ xy=2 \end{cases}$

x and y are the roots of

$$t^2-4t+2=0$$

Therefore, $t=2\pm\sqrt{2}$

Ans. $\begin{cases} \boldsymbol{x=2+\sqrt{2}} \\ \boldsymbol{y=2-\sqrt{2}} \end{cases}$ $\begin{cases} \boldsymbol{x=2-\sqrt{2}} \\ \boldsymbol{y=2+\sqrt{2}} \end{cases}$

Alternative method for example: From ①, $y=9-x$. We can substitute this into ②, and solve as usual. However, with terms such as $x+y$ and xy, it is easier to use the ***Root-Coefficient Relationships***.

J147b

(2) $\begin{cases} x+y=5 \\ xy=-6 \end{cases}$

x and y are the roots of

$t^2-5t-6=0$

$(t-6)(t+1)=0$

Threrefore,

$t=6,\ -1$

Ans. $\begin{cases} \boldsymbol{x=6} \\ \boldsymbol{y=-1} \end{cases}$ $\begin{cases} \boldsymbol{x=-1} \\ \boldsymbol{y=6} \end{cases}$

(3) $\begin{cases} x+y=\sqrt{3}+1 \\ xy=\sqrt{3} \end{cases}$

x and y are the roots of

$t^2-(\sqrt{3}+1)t+\sqrt{3}=0$

$(t-1)(t-\sqrt{3})=0$

Threrefore,

$t=1,\sqrt{3}$

Ans. $\begin{cases} \boldsymbol{x=1} \\ \boldsymbol{y=\sqrt{3}} \end{cases}$ $\begin{cases} \boldsymbol{x=\sqrt{3}} \\ \boldsymbol{y=1} \end{cases}$

J 148a KUMON

Simultaneous Equations

Solve the following equations.

Ex.

$$\begin{cases} x+y=4 & \cdots\cdots ① \\ x^2+xy+y^2=13 & \cdots\cdots ② \end{cases}$$

[Sol] From ②,

$$(x+y)^2-xy=13 \cdots\cdots ③$$

Substituting ① into ③,

$$16-xy=13$$

$$xy=3 \qquad \cdots\cdots ④$$

From ① and ④, x and y are the roots of

$$t^2-4t+3=0$$

$$(t-3)(t-1)=0$$

Therefore, $t=3, 1$

Ans. $\begin{cases} x=3 \\ y=1 \end{cases}$ $\begin{cases} x=1 \\ y=3 \end{cases}$

(1) $\begin{cases} x+y=4 & \cdots\cdots ① \\ x^2-xy+y^2=22 & \cdots\cdots ② \end{cases}$

From ②,

$$(x+y)^2-3xy=22 \quad \cdots\cdots ③$$

Substituting ① into ③,

$$16-3xy=22$$

$$xy=-2 \qquad \cdots\cdots ④$$

From ① and ④, x and y are the roots of

$$t^2-4t-2=0$$

Therefore, $t=2\pm\sqrt{6}$

Ans. $\begin{cases} \boldsymbol{x=2+\sqrt{6}} \\ \boldsymbol{y=2-\sqrt{6}} \end{cases}$ $\begin{cases} \boldsymbol{x=2-\sqrt{6}} \\ \boldsymbol{y=2+\sqrt{6}} \end{cases}$

J148b

(2) $\begin{cases} xy=3 & \cdots\cdots ① \\ x^2+y^2=10 & \cdots\cdots ② \end{cases}$

There are 4 solutions.

[Sol] From ②,

$$(x+y)^2-2xy=10 \quad \cdots\cdots ③$$

Substituting ① into ③,

$$(x+y)^2-6=10$$

$$(x+y)^2=16$$

$$x+y=\pm 4 \quad \cdots\cdots ④$$

From ① and ④, x and y are the roots of

$t^2-4t+3=0$	$t^2+4t+3=0$
$(t-3)(t-1)=0$	$(t+3)(t+1)=0$
Therefore, $t=3,\ 1$	Therefore, $t=-3,\ -1$

Ans. $\begin{cases} \boldsymbol{x=3} \\ \boldsymbol{y=1} \end{cases}$ $\begin{cases} \boldsymbol{x=1} \\ \boldsymbol{y=3} \end{cases}$ $\begin{cases} \boldsymbol{x=-3} \\ \boldsymbol{y=-1} \end{cases}$ $\begin{cases} \boldsymbol{x=-1} \\ \boldsymbol{y=-3} \end{cases}$

(3) $\begin{cases} xy=2 & \cdots\cdots ① \\ x^2+y^2=4 & \cdots\cdots ② \end{cases}$

From ②,

$$(x+y)^2-2xy=4 \quad \cdots\cdots ③$$

Substituting ① into ③,

$$(x+y)^2-4=4$$

$$(x+y)^2=8$$

$$x+y=\pm 2\sqrt{2} \quad \cdots\cdots ④$$

From ① and ④, x and y are the roots of

$t^2-2\sqrt{2}t+2=0$	$t^2+2\sqrt{2}t+2=0$
$(t-\sqrt{2})^2=0$	$(t+\sqrt{2})^2=0$
Therefore, $t=\sqrt{2}$	Therefore, $t=-\sqrt{2}$

Ans. $\begin{cases} \boldsymbol{x=\sqrt{2}} \\ \boldsymbol{y=\sqrt{2}} \end{cases}$ $\begin{cases} \boldsymbol{x=-\sqrt{2}} \\ \boldsymbol{y=-\sqrt{2}} \end{cases}$

J 149a KUMON

Simultaneous Equations

Solve the following equations.

(1) $\begin{cases} x^2+9xy+y^2=23 & \cdots\cdots ① \\ xy+x+y=5 & \cdots\cdots ② \end{cases}$

[Sol] Assuming that $x+y=A$, $xy=B$.

From ① and ②,

$$\begin{cases} A^2+7B=23 & \cdots\cdots ③ \\ A+B=5 & \cdots\cdots ④ \end{cases}$$

$x^2+9xy+y^2$
$=x^2+2xy+y^2+7xy$
$=A^2+7B$

From ④, $B=5-A$

Substituting this into ③,

$$A^2+7(\ 5-A\ )=23$$

$$A^2-\boxed{7}A+\boxed{12}=0$$

$$(\ A-4\ )(\ A-3\ )=0$$

When $A=\boxed{4}$, $B=\boxed{1}$

When $A=\boxed{3}$, $B=\boxed{2}$

(i) When $\begin{cases} x+y=\boxed{4} \\ xy=\boxed{1} \end{cases}$

x and y are the roots of

$$t^2-4t+1=0$$

Therefore, $t=2\pm\sqrt{3}$

(ii) When $\begin{cases} x+y=\boxed{3} \\ xy=\boxed{2} \end{cases}$

x and y are the roots of

$$t^2-3t+2=0$$

$$(t-2)(t-1)=0$$

Therefore, $t=2, 1$

Ans. $\begin{cases} \boldsymbol{x=2+\sqrt{3}} \\ \boldsymbol{y=2-\sqrt{3}} \end{cases}$ $\begin{cases} \boldsymbol{x=2-\sqrt{3}} \\ \boldsymbol{y=2+\sqrt{3}} \end{cases}$ $\begin{cases} \boldsymbol{x=2} \\ \boldsymbol{y=1} \end{cases}$ $\begin{cases} \boldsymbol{x=1} \\ \boldsymbol{y=2} \end{cases}$

J 149b

(2) $\begin{cases} xy = 12 & \cdots\cdots① \\ x^2 + y^2 = 25 & \cdots\cdots② \end{cases}$

Assuming that $x + y = A$, $xy = B$.

From ① and ②,

$\begin{cases} B = 12 & \cdots\cdots③ \\ A^2 - 2B = 25 & \cdots\cdots④ \end{cases}$

$x^2 + y^2 = (x+y)^2 - 2xy$
$= A^2 - 2B$

Substituting ③ into ④,

$A^2 - 24 = 25$

$A = \pm 7$

$\begin{cases} A = \pm 7 \\ B = 12 \end{cases}$

(i) When $\begin{cases} x + y = 7 \\ xy = 12 \end{cases}$

x and y are the roots of

$t^2 - 7t + 12 = 0$

$(t-4)(t-3) = 0$

Therefore, $t = 4, 3$

(ii) When $\begin{cases} x + y = -7 \\ xy = 12 \end{cases}$

x and y are the roots of

$t^2 + 7t + 12 = 0$

$(t+4)(t+3) = 0$

Therefore, $t = -4, -3$

Ans. $\begin{cases} \boldsymbol{x = 4} \\ \boldsymbol{y = 3} \end{cases}$ $\begin{cases} \boldsymbol{x = 3} \\ \boldsymbol{y = 4} \end{cases}$ $\begin{cases} \boldsymbol{x = -4} \\ \boldsymbol{y = -3} \end{cases}$ $\begin{cases} \boldsymbol{x = -3} \\ \boldsymbol{y = -4} \end{cases}$

J 150a KUMON

Simultaneous Equations

Solve the following equations.

(1) $\begin{cases} x-y=3 & \cdots\cdots ① \\ x^2+y^2=9 & \cdots\cdots ② \end{cases}$

From ①,

$y=x-3 \quad \cdots\cdots ③$

Substituting ③ into ②,

$x^2+(x-3)^2=9$

$2x^2-6x=0$

$x(x-3)=0$

$x=0,\ 3$

Substituting $x=0$ into ③,

$y=-3$

Substituting $x=3$ into ③,

$y=0$

Ans. $\begin{cases} \boldsymbol{x=0} \\ \boldsymbol{y=-3} \end{cases}$ $\begin{cases} \boldsymbol{x=3} \\ \boldsymbol{y=0} \end{cases}$

(2) $\begin{cases} x^2+y^2-2x+2y=3 & \cdots\cdots ① \\ x^2+y^2-4x+4y=1 & \cdots\cdots ② \end{cases}$

From ①−②,

$2x-2y=2$

$x=y+1 \quad \cdots\cdots ③$

Substituting ③ into ①,

$(y+1)^2+y^2-2(y+1)+2y=3$

$2y^2+2y-1=3$

$y^2+y-2=0$

$(y+2)(y-1)=0$

$y=-2,\ 1$

Substituting $y=-2$ into ③,

$x=-1$

Substituting $y=1$ into ③,

$x=2$

Ans. $\begin{cases} \boldsymbol{x=-1} \\ \boldsymbol{y=-2} \end{cases}$ $\begin{cases} \boldsymbol{x=2} \\ \boldsymbol{y=1} \end{cases}$

J150b

(3) $\begin{cases}(x-y)(x+2y)=0 & \cdots\cdots① \\ x^2-2xy+4y^2=12 & \cdots\cdots②\end{cases}$

From ①, $x=y$ or $x=-2y$

Substituting $x=y$ into ②,

$y^2-2y^2+4y^2=12$

$y^2=4$

$y=\pm 2$

Since $x=y$,

$x=\pm 2$

Substituting $x=-2y$ into ②,

$4y^2+4y^2+4y^2=12$

$y^2=1$

$y=\pm 1$

Since $x=-2y$,

$x=\mp 2$

Ans. $\begin{cases}\boldsymbol{x=\pm 2} \\ \boldsymbol{y=\pm 2}\end{cases}$ $\begin{cases}\boldsymbol{x=\pm 2} \\ \boldsymbol{y=\mp 1}\end{cases}$

(4) $\begin{cases}x+y=5 & \cdots\cdots① \\ x^3+y^3=35 & \cdots\cdots②\end{cases}$

From ②,

$(x+y)^3-3xy(x+y)=35 \cdots\cdots③$

Substituting ① into ③,

$125-15xy=35$

$xy=6 \qquad \cdots\cdots④$

From ① and ④, x and y are the roots of

$t^2-5t+6=0$

$(t-3)(t-2)=0$

Therefore,

$t=3,\ 2$

Ans. $\begin{cases}\boldsymbol{x=3} \\ \boldsymbol{y=2}\end{cases}$ $\begin{cases}\boldsymbol{x=2} \\ \boldsymbol{y=3}\end{cases}$

J151a KUMON

Dividing Polynomials

Divide the following polynomials.

Ex.

$$
\begin{array}{r@{\,}l}
 & \phantom{x^2+{}}x+2 \\
x+3 \big) & \overline{x^2+5x+6} \\
 & \underline{x^2+3x} \\
 & \phantom{x^2+{}}2x+6 \\
 & \phantom{x^2+{}}\underline{2x+6} \\
 & \phantom{x^2+2x+{}}0
\end{array}
$$

$(x+3)\times x$

$(x+3)\times 2$

(1)

$$
\begin{array}{r@{\,}l}
 & \boldsymbol{x+3} \\
x+5 \big) & \overline{x^2+8x+15} \\
 & \underline{x^2+5x} \\
 & \phantom{x^2+{}}3x+15 \\
 & \phantom{x^2+{}}\underline{3x+15} \\
 & 0
\end{array}
$$

(2)

$$
\begin{array}{r@{\,}l}
 & \boldsymbol{x+3} \\
2x-3 \big) & \overline{2x^2+3x-9} \\
 & \underline{2x^2-3x} \\
 & \phantom{2x^2+{}}6x-9 \\
 & \phantom{2x^2+{}}\underline{6x-9} \\
 & \phantom{2x^2+6x-{}}0
\end{array}
$$

(3)

$$
\begin{array}{r@{\,}l}
 & \boldsymbol{2x+5y} \\
3x-2y \big) & \overline{6x^2+11xy-10y^2} \\
 & \underline{6x^2-4xy} \\
 & \phantom{6x^2+{}}15xy-10y^2 \\
 & \phantom{6x^2+{}}\underline{15xy-10y^2} \\
 & 0
\end{array}
$$

(4)

$$
\begin{array}{r@{\,}l}
 & \boldsymbol{2x-3y} \\
3x+4y \big) & \overline{6x^2-xy-12y^2} \\
 & \underline{6x^2+8xy} \\
 & -9xy-12y^2 \\
 & \underline{-9xy-12y^2} \\
 & 0
\end{array}
$$

J151b

Ex.

$$
\begin{array}{r}
x^2+2x\ \ +3 \\
x^2-2x+2\,\overline{\big)\,x^4+0x^3+\ \ x^2-2x+6} \\
\underline{x^4-2x^3+2x^2} \\
2x^3-\ \ x^2-2x \\
\underline{2x^3-4x^2+4x} \\
3x^2-6x+6 \\
\underline{3x^2-6x+6} \\
0
\end{array}
$$

Leave a space as there is no x^3 term.

You can either leave a space or write $+0x^3$.

(5)

$$
\begin{array}{r}
\mathbf{4x^2+2x\ \ \ -3} \\
x^2+2x+3\,\overline{\big)\,4x^4+10x^3+13x^2+0x-9} \\
\underline{4x^4+\ \ 8x^3+12x^2} \\
2x^3+\ \ \ \ x^2+0x \\
\underline{2x^3+\ \ 4x^2+6x} \\
-\ \ 3x^2-6x-9 \\
\underline{-\ \ 3x^2-6x-9} \\
0
\end{array}
$$

(6)

$$
\begin{array}{r}
\mathbf{x^2+2x\ \ +3} \\
x^2-2x+2\,\overline{\big)\,x^4+0x^3+\ \ x^2-2x+6} \\
\underline{x^4-2x^3+2x^2} \\
2x^3-\ \ x^2-2x \\
\underline{2x^3-4x^2+4x} \\
3x^2-6x+6 \\
\underline{3x^2-6x+6} \\
0
\end{array}
$$

J 152a

KUMON

Dividing Polynomials

Divide the following polynomials.

(1) $(x^4+4x^2+16)\div(x^2-2x+4)$

$$
\begin{array}{r}
x^2+2x+4 \\
x^2-2x+4\,\overline{)\,x^4+0x^3+4x^2+0x+16} \\
\underline{x^4-2x^3+4x^2} \\
2x^3+0x^2+0x \\
\underline{2x^3-4x^2+8x} \\
4x^2-8x+16 \\
\underline{4x^2-8x+16} \\
0
\end{array}
$$

Leave spaces as there are no x^3 or x terms. Alternatively, write in $+0x^3$ and $+0x$.

Ans. $\boldsymbol{x^2+2x+4}$

(2) $(x^3+8)\div(x+2)$

$$
\begin{array}{r}
x^2-2x+4 \\
x+2\,\overline{)\,x^3+0x^2+0x+8} \\
\underline{x^3+2x^2} \\
-2x^2+0x \\
\underline{-2x^2-4x} \\
4x+8 \\
\underline{4x+8} \\
0
\end{array}
$$

Ans. $\boldsymbol{x^2-2x+4}$

J152b

(3)　$(a^4+a^2b^2+b^4)\div(a^2+ab+b^2)$

$$
\begin{array}{r|l}
 & a^2-ab+b^2 \\ \hline
a^2+ab+b^2 & a^4+0a^3b+a^2b^2+0ab^3+b^4 \\
 & \underline{a^4+a^3b+a^2b^2} \\
 & \quad -a^3b+0a^2b^2+0ab^3 \\
 & \quad \underline{-a^3b-a^2b^2-ab^3} \\
 & \qquad\qquad a^2b^2+ab^3+b^4 \\
 & \qquad\qquad \underline{a^2b^2+ab^3+b^4} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

Ans.　$\boldsymbol{a^2-ab+b^2}$

(4)　$(x^4-y^4)\div(x-y)$

$$
\begin{array}{r|l}
 & x^3+x^2y+xy^2+y^3 \\ \hline
x-y & x^4+0x^3y+0x^2y^2+0xy^3-y^4 \\
 & \underline{x^4-x^3y} \\
 & \quad x^3y+0x^2y^2 \\
 & \quad \underline{x^3y-x^2y^2} \\
 & \qquad x^2y^2+0xy^3 \\
 & \qquad \underline{x^2y^2-xy^3} \\
 & \qquad\qquad xy^3-y^4 \\
 & \qquad\qquad \underline{xy^3-y^4} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

Ans.　$\boldsymbol{x^3+x^2y+xy^2+y^3}$

J 153a KUMON

Dividing Polynomials

Divide the following polynomials.

Ex.

$(2x^3+5x^2+9)\div(2x^2+x+3)$

[Sol]

$$
\begin{array}{r l}
 & \;x \quad +2 \\
2x^2+x+3 & \overline{)\,2x^3+5x^2+0x+9} \\
 & \;\;\underline{2x^3+x^2+3x} \\
 & \qquad\quad 4x^2-3x+9 \\
 & \qquad\quad \underline{4x^2+2x+6} \\
 & \qquad\qquad\quad -5x+3
\end{array}
$$

Ans. Quotient $x+2$ Remainder $-5x+3$

(1) $(2x^4-18x^2+3x-4)\div(x^3-3x^2-2)$

$$
\begin{array}{r l}
 & \;2x \quad +6 \\
x^3-3x^2-2 & \overline{)\,2x^4+0x^3-18x^2+3x-4} \\
 & \;\;\underline{2x^4-6x^3 \qquad\qquad -4x} \\
 & \qquad\quad 6x^3-18x^2+7x-4 \\
 & \qquad\quad \underline{6x^3-18x^2 \qquad\; -12} \\
 & \qquad\qquad\qquad\qquad 7x+8
\end{array}
$$

Ans. Quotient $\mathbf{2x+6}$ Remainder $\mathbf{7x+8}$

J153b

（2） $(2a^3+a^2-5a+2)\div(a+1)$

$$
\begin{array}{r}
2a^2-a\quad-4 \\
a+1\,\big)\,\overline{2a^3+\ a^2-5a+2} \\
\underline{2a^3+2a^2\qquad\quad} \\
-a^2-5a\quad \\
\underline{-a^2-\ a\quad} \\
-4a+2 \\
\underline{-4a-4} \\
6
\end{array}
$$

Ans. Quotient $\boldsymbol{2a^2-a-4}$　Remainder $\mathbf{6}$

（3） $(x^2-3xy+2y^2+3x-6y-8)\div(x-y+4)$

$$
\begin{array}{r}
x\ -2y\ \ -1\qquad\qquad\quad \\
x-y+4\,\big)\,\overline{x^2-3xy+2y^2+3x-6y-8} \\
\underline{x^2-\ xy\qquad\quad+4x\qquad\quad} \\
\underline{-2xy+2y^2-\ x-6y\quad} \\
\underline{-2xy+2y^2\qquad-8y\quad} \\
-\ x+2y-8 \\
\underline{-\ x+\ y-4} \\
y-4
\end{array}
$$

Ans. Quotient $\boldsymbol{x-2y-1}$　Remainder $\boldsymbol{y-4}$

J 154a KUMON

Dividing Polynomials

$17 \div 5 = 3$ Remainder 2 This relationship may be written as $17 = 5 \times 3 + 2$.	Similarly, $(x^3 - x^2 - 1) \div (x-2)$ $= x^2 + x + 2$ Remainder 3 This relationship may be written as $x^3 - x^2 - 1 = (x-2)(x^2+x+2) + 3$.

1. Complete the following exercises.

(1) Divide $x^3 + 4x^2 - 5x + 3$ by $x - 2$.

$$\begin{array}{r l}
 & x^2 + 6x + 7 \\ \cline{2-2}
x-2 \big) & x^3 + 4x^2 - 5x + 3 \\
 & x^3 - 2x^2 \\ \cline{2-2}
 & \qquad 6x^2 - 5x \\
 & \qquad 6x^2 - 12x \\ \cline{2-2}
 & \qquad\qquad 7x + 3 \\
 & \qquad\qquad 7x - 14 \\ \cline{2-2}
 & \qquad\qquad\quad 17
\end{array}$$

Note:
For $26 \div 7 = 3\ R5$
7 is the divisor,
3 is the quotient,
5 is the remainder.

Ans. Quotient $\boldsymbol{x^2 + 6x + 7}$ Remainder **17**

(2) Using the result of (1), write the relationship between $x^3 + 4x^2 - 5x + 3$ and $x - 2$.

Ans. $x^3 + 4x^2 - 5x + 3 = \boldsymbol{(x-2)(x^2+6x+7) + 17}$

J 154b

2. In each question, divide A by B, and write the relationship between the two.

(1) $A = 2x^3 + x^2 - 5x + 2 \quad B = x + 1$

$$
\begin{array}{r|l}
 & 2x^2 - x \quad - 4 \\
\hline
x+1 & 2x^3 + x^2 - 5x + 2 \\
 & \underline{2x^3 + 2x^2} \\
 & -x^2 - 5x \\
 & \underline{-x^2 - x} \\
 & -4x + 2 \\
 & \underline{-4x - 4} \\
 & 6
\end{array}
$$

Ans. $A = B(\ \mathbf{2x^2 - x - 4}\ ) + \boxed{\mathbf{6}}$

(2) $A = x^3 - 3x^2 + 4x + 30 \quad B = x - 2$

$$
\begin{array}{r|l}
 & x^2 - x \quad + 2 \\
\hline
x-2 & x^3 - 3x^2 + 4x + 30 \\
 & \underline{x^3 - 2x^2} \\
 & -x^2 + 4x \\
 & \underline{-x^2 + 2x} \\
 & 2x + 30 \\
 & \underline{2x - 4} \\
 & 34
\end{array}
$$

Ans. $A = \mathbf{B(x^2 - x + 2) + 34}$

J155a KUMON

Dividing Polynomials

In each question, divide A by B, and write the relationship between the two.

(1) $\boldsymbol{A}=x^4+x^3-8x^2-9x+9 \qquad \boldsymbol{B}=x-3$

$$
\begin{array}{r|l}
 & x^3+4x^2+4x+3 \\
\hline
x-3 & x^4+x^3-8x^2-9x+9 \\
 & \underline{x^4-3x^3} \\
 & \quad 4x^3-8x^2 \\
 & \quad \underline{4x^3-12x^2} \\
 & \qquad 4x^2-9x \\
 & \qquad \underline{4x^2-12x} \\
 & \qquad\quad 3x+9 \\
 & \qquad\quad \underline{3x-9} \\
 & \qquad\qquad 18
\end{array}
$$

Ans. $A=\boldsymbol{B(x^3+4x^2+4x+3)+18}$

(2) $\boldsymbol{A}=2x^4+25x^3+4x+10 \qquad \boldsymbol{B}=2x+1$

$$
\begin{array}{r|l}
 & x^3+12x^2-6x+5 \\
\hline
2x+1 & 2x^4+25x^3+0x^2+4x+10 \\
 & \underline{2x^4+x^3} \\
 & \quad 24x^3+0x^2 \\
 & \quad \underline{24x^3+12x^2} \\
 & \qquad -12x^2+4x \\
 & \qquad \underline{-12x^2-6x} \\
 & \qquad\quad 10x+10 \\
 & \qquad\quad \underline{10x+5} \\
 & \qquad\qquad 5
\end{array}
$$

Ans. $A=\boldsymbol{B(x^3+12x^2-6x+5)+5}$

J155b

（３）　$A=x^4+6x^3-7x^2+3x+9$　　$B=x^2+3x-2$

$$
\begin{array}{r}
x^2+3x-14 \\
x^2+3x-2\,\overline{)\,x^4+6x^3-7x^2+3x+9} \\
\underline{x^4+3x^3-2x^2} \\
3x^3-5x^2+3x \\
\underline{3x^3+9x^2-6x} \\
-14x^2+9x+9 \\
\underline{-14x^2-42x+28} \\
51x-19
\end{array}
$$

Ans.　$A=\boldsymbol{B(x^2+3x-14)+51x-19}$

（４）　$A=x^3+6x^2+6x-6$　　$B=x^2+3x-3$

$$
\begin{array}{r}
x+3 \\
x^2+3x-3\,\overline{)\,x^3+6x^2+6x-6} \\
\underline{x^3+3x^2-3x} \\
3x^2+9x-6 \\
\underline{3x^2+9x-9} \\
3
\end{array}
$$

Ans.　$A=\boldsymbol{B(x+3)+3}$

J 156a KUMON

Dividing Polynomials

Ex.

When A is divided by $2x^2-1$, the quotient is $2x-1$ with remainder $x-2$. Find A.

[Sol] $A=(2x^2-1)(2x-1)+x-2$ ☜ Refer to J 154a.

Therefore,

$A=4x^3-2x^2-x-1$

1. When A is divided by $x+1$, the quotient is x^2+2 with remainder 3. Find A.

[Sol] $A=(x+1)(x^2+2)+3$

Therefore,

$\boldsymbol{A}=x^3+2x+x^2+2+3$

$\boldsymbol{=x^3+x^2+2x+5}$

2. When A is divided by $2x+3$, the quotient is x^2-2x+1 with remainder -2. Find A.

[Sol] $A=(2x+3)(x^2-2x+1)-2$

Therefore,

$\boldsymbol{A}=2x^3-4x^2+2x+3x^2-6x+3-2$

$\boldsymbol{=2x^3-x^2-4x+1}$

J156b

3. When A is divided by x^2+1, the quotient is x^2+1 with remainder $x+1$. Find A.

[Sol] $A=(x^2+1)(x^2+1)+x+1$
Therefore,
$\boldsymbol{A}=x^4+2x^2+1+x+1$
$\boldsymbol{=x^4+2x^2+x+2}$

4. When A is divided by x^2+x+1, the quotient is x^2+1 with remainder $2x$. Find A.

[Sol] $A=(x^2+x+1)(x^2+1)+2x$
Therefore,
$\boldsymbol{A}=x^4+x^2+x^3+x+x^2+1+2x$
$\boldsymbol{=x^4+x^3+2x^2+3x+1}$

J157a KUMON

Dividing Polynomials

Ex.

When x^3+x^2+4x-1 is divided by B, the quotient is $x+1$ with remainder $3x-2$. Find B.

[Sol] $x^3+x^2+4x-1=B(x+1)+3x-2$

$x^3+x^2+x+1=B(x+1)$

Move $3x-2$ to the left side.

Therefore,

$B=(x^3+x^2+x+1)\div(x+1)$

$$\begin{array}{r|l} & x^2 \qquad +1 \\ \hline x+1 & x^3+x^2+x+1 \\ & \underline{x^3+x^2} \\ & \qquad\quad x+1 \\ & \qquad\quad \underline{x+1} \\ & \qquad\qquad 0 \end{array}$$

Ans. $B=x^2+1$

1. When $2x^3+x^2-x+1$ is divided by B, the quotient is $2x+3$ with remainder -2. Find B.

[Sol] $2x^3+x^2-x+1=B(2x+3)-2$

$2x^3+x^2-x+3=B(2x+3)$

Therefore,

$B=(2x^3+x^2-x+3)\div(2x+3)$

$$\begin{array}{r|l} & x^2-x \quad +1 \\ \hline 2x+3 & 2x^3+x^2-x+3 \\ & \underline{2x^3+3x^2} \\ & \quad -2x^2-x \\ & \quad \underline{-2x^2-3x} \\ & \qquad\qquad 2x+3 \\ & \qquad\qquad \underline{2x+3} \\ & \qquad\qquad\quad 0 \end{array}$$

Ans. $\boldsymbol{B=x^2-x+1}$

Ex.

When $2x^3-x^2-ax+b$ is divided by x^2-2x+1, the remainder is $3x+2$. Find the values of a and b.

[Sol]

$$
\begin{array}{r|l}
 & 2x+3 \\ \cline{2-2}
x^2-2x+1 & 2x^3-x^2-ax+b \\
 & 2x^3-4x^2+2x \\ \cline{2-2}
 & 3x^2-(a+2)x+b \\
 & 3x^2-6x+3 \\ \cline{2-2}
 & (-a+4)x+b-3
\end{array}
$$

Since the remainder is $3x+2$,

$$(-a+4)x+b-3=3x+2$$

Comparing left and right hand sides,

$$\begin{cases} -a+4=3 & \cdots ① \\ b-3=2 & \cdots ② \end{cases}$$

Match the coefficients of x on both sides, and the constant terms on both sides.

From ① and ②,

$$a=1,\ b=5$$

2. When x^4+ax^2+b is divided by x^2+x+1, the remainder is $-2x+2$. Find the values of a and b.

[Sol]

$$
\begin{array}{r|l}
 & x^2-x+a \\ \cline{2-2}
x^2+x+1 & x^4+0x^3+ax^2+0x+b \\
 & x^4+x^3+x^2 \\ \cline{2-2}
 & -x^3+(a-1)x^2+0x \\
 & -x^3-x^2-x \\ \cline{2-2}
 & ax^2+x+b \\
 & ax^2+ax+a \\ \cline{2-2}
 & (1-a)x+b-a
\end{array}
$$

Since the remainder is $-2x+2$,

$$(1-a)x+b-a=-2x+2$$

Comparing left and right hand sides,

$$\begin{cases} 1-a=-2 & \cdots ① \\ b-a=2 & \cdots ② \end{cases}$$

From ① and ②,

$$\boldsymbol{a=3,\ b=5}$$

J 158a KUMON

Dividing Polynomials

1. Simplify the expressions.

Ex.

$$\frac{4x-5}{x-2}=4+\frac{3}{x-2}$$

$$\begin{array}{r} 4 \\ x-2\,\overline{)\,4x-5} \\ \underline{4x-8} \\ 3 \end{array}$$

(1) $\dfrac{2x-1}{x-2}=\mathbf{2+\dfrac{3}{x-2}}$

$$\begin{array}{r} 2 \\ x-2\,\overline{)\,2x-1} \\ \underline{2x-4} \\ 3 \end{array}$$

(2) $\dfrac{-2x+1}{x+3}=\mathbf{-2+\dfrac{7}{x+3}}$

$$\begin{array}{r} -2 \\ x+3\,\overline{)\,-2x+1} \\ \underline{-2x-6} \\ 7 \end{array}$$

(3) $\dfrac{4x+5}{2x-1}=\mathbf{2+\dfrac{7}{2x-1}}$

$$\begin{array}{r} 2 \\ 2x-1\,\overline{)\,4x+5} \\ \underline{4x-2} \\ 7 \end{array}$$

Note: You can also simplify in this way:

$$\frac{4x-5}{x-2}=\frac{4(x-2)+3}{x-2}=\frac{4(x-2)}{x-2}+\frac{3}{x-2}=4+\frac{3}{x-2}$$

J158b

2. Simplify the expressions.

Ex.

$$\frac{x^2-3x+3}{x-1}=x-2+\frac{1}{x-1}$$

$$\begin{array}{r|l} & x\ -2 \\ \cline{2-2} x-1 & x^2-3x+3 \\ & \underline{x^2-\ \ x} \\ & -2x+3 \\ & \underline{-2x+2} \\ & 1 \end{array}$$

(1) $\dfrac{x^2-x+4}{x-2}=\boldsymbol{x+1+\dfrac{6}{x-2}}$

$$\begin{array}{r|l} & x\ +1 \\ \cline{2-2} x-2 & x^2-\ \ x+4 \\ & \underline{x^2-2x} \\ & x+4 \\ & \underline{x-2} \\ & 6 \end{array}$$

(2) $\dfrac{2x^2-x+1}{x+1}=\boldsymbol{2x-3+\dfrac{4}{x+1}}$

$$\begin{array}{r|l} & 2x\ -3 \\ \cline{2-2} x+1 & 2x^2-\ \ x+1 \\ & \underline{2x^2+2x} \\ & -3x+1 \\ & \underline{-3x-3} \\ & 4 \end{array}$$

(3) $\dfrac{2x^2-3x+2}{2x+1}=\boldsymbol{x-2+\dfrac{4}{2x+1}}$

$$\begin{array}{r|l} & x\ -2 \\ \cline{2-2} 2x+1 & 2x^2-3x+2 \\ & \underline{2x^2+\ \ x} \\ & -4x+2 \\ & \underline{-4x-2} \\ & 4 \end{array}$$

J 159a KUMON

Dividing Polynomials

Simplify the following expressions.

Ex.

$$\frac{a-4}{a-5}-\frac{a-5}{a-6}=\left(1+\frac{1}{a-5}\right)-\left(1+\frac{1}{a-6}\right)$$

$$=\frac{1}{a-5}-\frac{1}{a-6}=-\frac{1}{(a-5)(a-6)}$$

Note: Simplifying the fractions first makes the calculations easier.

(1) $$\frac{x-8}{x-4}+\frac{x-5}{x-7}-2=\left(1+\frac{-4}{x-4}\right)+\left(1+\frac{2}{x-7}\right)-2$$

$$=\frac{-4}{x-4}+\frac{2}{x-7}$$

$$=\frac{-4(x-7)+2(x-4)}{(x-4)(x-7)}$$

$$=\boldsymbol{\frac{-2x+20}{(x-4)(x-7)}}$$

$$\left[=-\boldsymbol{\frac{2(x-10)}{(x-4)(x-7)}}\right]$$

J159b

(2) $\dfrac{2a}{a-1}-\dfrac{a-1}{a+1}-\dfrac{a+1}{a}=\left(2+\dfrac{2}{a-1}\right)-\left(1+\dfrac{-2}{a+1}\right)-\left(1+\dfrac{1}{a}\right)$

$$=\frac{2}{a-1}-\frac{-2}{a+1}-\frac{1}{a}$$

$$=\frac{2(a+1)+2(a-1)}{(a-1)(a+1)}-\frac{1}{a}$$

$$=\frac{4a}{(a-1)(a+1)}-\frac{1}{a}$$

$$=\frac{4a^2-(a^2-1)}{a(a-1)(a+1)}$$

$$=\boldsymbol{\frac{3a^2+1}{a(a-1)(a+1)}}$$

(3) $\dfrac{x+2}{x+1}-\dfrac{x+3}{x+2}-\dfrac{x+5}{x+4}+\dfrac{x+6}{x+5}$

$$=\left(1+\frac{1}{x+1}\right)-\left(1+\frac{1}{x+2}\right)-\left(1+\frac{1}{x+4}\right)+\left(1+\frac{1}{x+5}\right)$$

$$=\frac{(x+2)-(x+1)}{(x+1)(x+2)}-\frac{(x+5)-(x+4)}{(x+4)(x+5)}$$

$$=\frac{1}{(x+1)(x+2)}-\frac{1}{(x+4)(x+5)}$$

$$=\frac{(x^2+9x+20)-(x^2+3x+2)}{(x+1)(x+2)(x+4)(x+5)}$$

$$=\boldsymbol{\frac{6x+18}{(x+1)(x+2)(x+4)(x+5)}}$$

$$\left[=\boldsymbol{\frac{6(x+3)}{(x+1)(x+2)(x+4)(x+5)}}\right]$$

J 160a KUMON

Dividing Polynomials

1. Divide the following polynomial.

$$(x^4+2x^3-12x^2-4)\div(x^2-2x-6)$$

[Sol]

$$\begin{array}{r l}
 & x^2+4x+2 \\ \cline{2-2}
x^2-2x-6 \,\big) & x^4+2x^3-12x^2+0x-4 \\
 & x^4-2x^3-6x^2 \\ \cline{2-2}
 & \quad 4x^3-6x^2+0x \\
 & \quad 4x^3-8x^2-24x \\ \cline{2-2}
 & \qquad 2x^2+24x-4 \\
 & \qquad 2x^2-4x-12 \\ \cline{2-2}
 & \qquad\qquad 28x+8
\end{array}$$

Ans. Quotient $\boldsymbol{x^2+4x+2}$ Remainder $\boldsymbol{28x+8}$

2. When $x^3+ax^2-4x-a^2$ is divided by x^2-x-a, the remainder is $x+b$. Find the values of a and b.

[Sol]

$$\begin{array}{r l}
 & x+(a+1) \\ \cline{2-2}
x^2-x-a \,\big) & x^3+ax^2-4x-a^2 \\
 & x^3-x^2-ax \\ \cline{2-2}
 & \quad (a+1)x^2+(a-4)x-a^2 \\
 & \quad (a+1)x^2-(a+1)x-a(a+1) \\ \cline{2-2}
 & \qquad (2a-3)x+a
\end{array}$$

Since the remainder is $x+b$,

$$(2a-3)x+a=x+b$$

Comparing left and right hand sides,

$$\begin{cases} 2a-3=1 & \cdots① \\ a=b & \cdots② \end{cases}$$

From ① and ②,

$\boldsymbol{a=2,\ b=2}$

J 160b

3. When A is divided by x^2+x+1, the quotient is x^2-2 with remainder $2x+7$. Find the quotient and the remainder when A is divided by x^2-2x+4.

[Sol] $A=(x^2+x+1)(x^2-2)+2x+7$

$=x^4-2x^2+x^3-2x+x^2-2+2x+7$

$=x^4+x^3-x^2+5$

Dividing A by x^2-2x+4:

$$
\begin{array}{r|l}
 & x^2+3x+1 \\
\hline
x^2-2x+4 & x^4+x^3-x^2+0x+5 \\
 & x^4-2x^3+4x^2 \\
\cline{2-2}
 & 3x^3-5x^2+0x \\
 & 3x^3-6x^2+12x \\
\cline{2-2}
 & x^2-12x+5 \\
 & x^2-2x+4 \\
\cline{2-2}
 & -10x+1
\end{array}
$$

Ans. Quotient $\boldsymbol{x^2+3x+1}$ Remainder $\boldsymbol{-10x+1}$

J161a

Remainder Theorem

Ex.

Determine the value of a for which x^2+ax+3 is exactly divisible by $x-1$.
(i.e. the remainder is equal to zero)

[Sol]

$$
\begin{array}{r|l}
 & x+(a+1) \\
\hline
x-1 & x^2+ax+3 \\
 & \underline{x^2 - x} \\
 & (a+1)x+3 \\
 & \underline{(a+1)x-a-1} \\
 & a+4
\end{array}
$$

Therefore, $a+4=0$

$a=-4$

"To be divisible by" means the remainder is zero.

1. Determine the value of a for which x^2+2x-a is exactly divisible by $x+4$.

[Sol]

$$
\begin{array}{r|l}
 & x-2 \\
\hline
x+4 & x^2+2x-a \\
 & \underline{x^2+4x} \\
 & -2x-a \\
 & \underline{-2x-8} \\
 & -a+8
\end{array}
$$

Therefore, $-a+8=0$

$\boldsymbol{a=8}$

J161b

2. Determine the value of a for which $2x^2+7x+a$ is exactly divisible by $x+1$.

[Sol]

$$\begin{array}{r|l} & 2x + 5 \\ \hline x+1 & 2x^2+7x+a \\ & \underline{2x^2+2x} \\ & \qquad 5x+a \\ & \qquad \underline{5x+5} \\ & \qquad\quad a-5 \end{array}$$

Therefore, $a-5=0$

$$\boldsymbol{a=5}$$

3. Determine the value of a for which x^2-ax+6 is exactly divisible by $x-2$.

[Sol]

$$\begin{array}{r|l} & x + (-a+2) \\ \hline x-2 & x^2 \qquad -ax+6 \\ & \underline{x^2 \qquad -2x} \\ & \quad (-a+2)x+6 \\ & \quad \underline{(-a+2)x+2a-4} \\ & \qquad\qquad -2a+10 \end{array}$$

Therefore, $-2a+10=0$

$$\boldsymbol{a=5}$$

4. Determine the value of a for which x^2+2x-a is divided by $x+4$ with remainder 3.

[Sol]

$$\begin{array}{r|l} & x - 2 \\ \hline x+4 & x^2+2x-a \\ & \underline{x^2+4x} \\ & \quad -2x-a \\ & \quad \underline{-2x-8} \\ & \qquad -a+8 \end{array}$$

Therefore, $-a+8=3$

$$\boldsymbol{a=5}$$

J 162a KUMON

Remainder Theorem

Notation

- $P(x)$ and $Q(x)$ are used to represent polynomial expressions in x.
- When 2 is substituted for x, this is written as $P(2)$.

Complete the following exercises on $P(x)=x^3+x^2-5x+6$.

Ex.

First find $P(2)$, then find the quotient and remainder when $P(x)$ is divided by $x-2$.

[Sol] $P(2)=2^3+2^2-5\times2+6=\mathbf{8}$ ☜ Substitute $x=2$ into $P(x)=x^3+x^2-5x+6$.

$$\begin{array}{r|l} & x^2+3x+1 \\ \hline x-2 & x^3+x^2-5x+6 \\ & x^3-2x^2 \\ \hline & 3x^2-5x \\ & 3x^2-6x \\ \hline & x+6 \\ & x-2 \\ \hline & \mathbf{8} \end{array}$$

Ans. Quotient x^2+3x+1
Remainder **8**

▶ **When $P(x)$ is divided by $x-2$, the remainder is $P(2)$.**

(1) First find $P(3)$, then find the quotient and remainder when $P(x)$ is divided by $x-3$.

[Sol] $P(3)=\boxed{\mathbf{3}}^3+\boxed{\mathbf{3}}^2-5\times\boxed{\mathbf{3}}+6=\mathbf{27}$

$$\begin{array}{r|l} & x^2+4x+7 \\ \hline x-3 & x^3+x^2-5x+6 \\ & x^3-3x^2 \\ \hline & 4x^2-5x \\ & 4x^2-12x \\ \hline & 7x+6 \\ & 7x-21 \\ \hline & 27 \end{array}$$

Ans. Quotient $\mathbf{x^2+4x+7}$
Remainder **27**

▶ When $P(x)$ is divided by $x-3$, the remainder is $\boxed{P(\ \mathbf{3}\ )}$.

(2) First find $P(-2)$, then find the quotient and remainder when $P(x)$ is divided by $x+2$.

[Sol] $P(-2)=(-2)^3+(-2)^2-5\times(-2)+6=\mathbf{12}$

$$
\begin{array}{r}
x^2-x-3 \\
x+2\,\overline{)\,x^3+x^2-5x+6} \\
\underline{x^3+2x^2} \\
-x^2-5x \\
\underline{-x^2-2x} \\
-3x+6 \\
\underline{-3x-6} \\
12
\end{array}
$$

Ans. Quotient $\boldsymbol{x^2-x-3}$
Remainder **12**

▶ When $P(x)$ is divided by $\boxed{\boldsymbol{x+2}}$, the remainder is $P(-2)$.

(3) First find $P(-1)$, then find the quotient and remainder when $P(x)$ is divided by $x+1$.

[Sol] $P(-1)=(-1)^3+(-1)^2-5\times(-1)+6=\mathbf{11}$

$$
\begin{array}{r}
x^2-5 \\
x+1\,\overline{)\,x^3+x^2-5x+6} \\
\underline{x^3+x^2} \\
-5x+6 \\
\underline{-5x-5} \\
11
\end{array}
$$

Ans. Quotient $\boldsymbol{x^2-5}$
Remainder **11**

▶ When $P(x)$ is divided by $\boxed{\boldsymbol{x+1}}$, the remainder is $\boxed{P(\ \mathbf{-1}\ )}$.

- Looking at the example on side a, $P(x)=(x-2)(x^2+3x+1)+8$, and substitution gives $P(2)=8$.
- If a polynomial $P(x)$ is divided by $x-a$, we can express the relationship with the quotient $Q(x)$ and remainder R as:

  $$P(x)=\underline{(x-a)Q(x)}+R$$

- When $x=a$, the underlined portion of the above equation becomes $(a-a)Q(x)=0$. Thus, $P(a)=(a-a)Q(x)+R=R$

Conclusion: **When $P(x)$ is divided by $x-a$, the remainder is $P(a)$.**

J 163a KUMON

Remainder Theorem

The Remainder Theorem

When $P(x)$ is divided by $x-a$, the remainder is $P(a)$.

1. Use the *Remainder Theorem* to find the remainder when the given $P(x)$ is divided by the term on the right.

(1) $P(x)=x^3-7x^2+4x-3$ $\qquad x-2$

$$P(2)=2^3-7\times2^2+4\times2-3$$
$$=\mathbf{-15}$$

(2) $P(x)=2x^3-x+1$ $\qquad x+3$

$$P(-3)=2\times(-3)^3-(-3)+1$$
$$=\mathbf{-50}$$

$x-a-1=[x-(a+1)]$

(3) $P(x)=x^3-2ax^2+1$ $\qquad x-a-1$

$$P(a+1)=(a+1)^3-2a(a+1)^2+1$$
$$=a^3+3a^2+3a+1-2a(a^2+2a+1)+1$$
$$=\boldsymbol{-a^3-a^2+a+2}$$

(4) $P(x)=x^4-2x^2-1$ $\qquad 2x-1$

$$P\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^4-2\times\left(\frac{1}{2}\right)^2-1$$
$$=\frac{1}{16}-\frac{1}{2}-1=\mathbf{-\frac{23}{16}}$$

(5) $P(x)=x^5-6x^2+x+6$ $\qquad x+2$

$$P(-2)=(-2)^5-6\times(-2)^2+(-2)+6$$
$$=\mathbf{-52}$$

J163b

2. Find the remainders when $P(x)=2x^2-3x+4$ is divided by each of the following terms.

(1) $x-1$

$$\begin{aligned}P(1)&=2\times1^2-3\times1+4\\&=\mathbf{3}\end{aligned}$$

(2) $x+2$

$$\begin{aligned}P(-2)&=2\times(-2)^2-3\times(-2)+4\\&=\mathbf{18}\end{aligned}$$

(3) $3x-4$

$$\begin{aligned}P\left(\frac{4}{3}\right)&=2\times\left(\frac{4}{3}\right)^2-3\times\left(\frac{4}{3}\right)+4\\&=\mathbf{\frac{32}{9}}\end{aligned}$$

(4) $x+a$

$$\begin{aligned}P(-a)&=2(-a)^2-3(-a)+4\\&=\boldsymbol{2a^2+3a+4}\end{aligned}$$

(5) $x-a-1$ 　$x-a-1=[x-(a+1)]$

$$\begin{aligned}P(a+1)&=2(a+1)^2-3(a+1)+4\\&=2(a^2+2a+1)-3a-3+4\\&=\boldsymbol{2a^2+a+3}\end{aligned}$$

J 164a KUMON

Remainder Theorem

1. By using the *Remainder Theorem*, find the value of a so that $P(x)=2x^2+7x+a$ can be divided by $x+1$ with no remainder.

[Sol] $P(-1)=2\times(-1)^2+7\times(-1)+a$
$=a-5$
For there to be no remainder, $P(-1)=0$
$a-5=0$
$\boldsymbol{a=5}$

2. Find the value of a so that $P(x)=x^2-ax+6$ can be divided by $x-2$ with no remainder.

[Sol] $P(2)=4-2a+6$
$=-2a+10$
For there to be no remainder, $P(2)=0$
$-2a+10=0$
$\boldsymbol{a=5}$

3. Find the value of a so that $P(x)=a^2x^3-3ax+1$ can be divided by $x-2$ with no remainder.

[Sol] $P(2)=8a^2-6a+1$
For there to be no remainder, $P(2)=0$
$8a^2-6a+1=0$
$(4a-1)(2a-1)=0$
$\boldsymbol{a=\frac{1}{4},\ \frac{1}{2}}$

Note: Compare questions 1. and 2. above, with questions 2. and 3. on J161b.

The ***Remainder Theorem*** can only be used when dividing by a linear expression. When dividing by a quadratic expression, use long division.

Ex.

When x^3+6x^2+ax+b is divided by x^2+3x-2, the remainder is 3. Find the values of a and b.

[Sol]

$$
\begin{array}{r}
x+3 \\
x^2+3x-2\,\big)\overline{\,x^3+6x^2\qquad+ax+b} \\
\underline{x^3+3x^2\qquad-2x} \\
3x^2+(a+2)x+b \\
\underline{3x^2\qquad+9x-6} \\
(a-7)x+b+6
\end{array}
$$

Since the remainder is 3,

$$(a-7)x+b+6=3$$

$$\begin{cases} a-7=0 & \cdots① \\ b+6=3 & \cdots② \end{cases}$$

In order for the remainder to be 3, $a-7=0$ and $b+6=3$ must be true.

From ① and ②,

$$a=7,\ b=-3$$

4. Find the values of a and b for which x^3+2x^2+ax+b is divisible by x^2+3x-1.
(Remember that "x^3+2x^2+ax+b is divisible by x^2+3x-1" means that the remainder equals zero.)

[Sol]

$$
\begin{array}{r}
x-1 \\
x^2+3x-1\,\big)\overline{\,x^3+2x^2\qquad+ax+b} \\
\underline{x^3+3x^2\qquad-x} \\
-x^2+(a+1)x+b \\
\underline{-x^2\qquad-3x\quad+1} \\
(a+4)x+b-1
\end{array}
$$

Since the remainder is 0,

$$(a+4)x+b-1=0$$

$$\begin{cases} a+4=0 & \cdots① \\ b-1=0 & \cdots② \end{cases}$$

In order for the remainder to be 0, $a+4=0$ and $b-1=0$ must be true.

From ① and ②,

$$\boldsymbol{a=-4,\ b=1}$$

J 165a KUMON

Remainder Theorem

Ex.

Given that $P(x)$ is divided by $(x-2)(x-3)$, express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$,
and let the remainder be $ax+b$. ☜ Since we are dividing by a quadratic, we write the remainder as $ax+b$.

$$P(x)=(x-2)(x-3)Q(x)+ax+b$$

1. Given that $P(x)$ is divided by $(x-2)(x+3)$, express the relationship with the quotient and the remainder.

   [Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

   $$\boldsymbol{P(x)=(x-2)(x+3)Q(x)+ax+b}$$

2. Given that $P(x)$ is divided by $(x+1)(x+4)$, express the relationship with the quotient and the remainder.

   [Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

   $$\boldsymbol{P(x)=(x+1)(x+4)Q(x)+ax+b}$$

3. Given that $P(x)$ is divided by x^2-x-2, express the relationship with the quotient and the remainder.

   [Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

   $$\boldsymbol{P(x)=(x^2-x-2)Q(x)+ax+b}$$
   $$\boldsymbol{[=(x-2)(x+1)Q(x)+ax+b]}$$

Ex.

Given that $P(x)$ is divided by $(x-1)(x-2)(x-3)$, express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$, and let the remainder be ax^2+bx+c.

Since we are dividing by a cubic, we write the remainder as ax^2+bx+c.

$$P(x)=(x-1)(x-2)(x-3)Q(x)+ax^2+bx+c$$

4. Given that $P(x)$ is divided by $(x-1)(x-2)(x+3)$, express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$, and let the remainder be ax^2+bx+c.

$$\boldsymbol{P(x)=(x-1)(x-2)(x+3)Q(x)+ax^2+bx+c}$$

5. Given that $P(x)$ is divided by $(x+1)(x+2)(x+3)$, express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$, and let the remainder be ax^2+bx+c.

$$\boldsymbol{P(x)=(x+1)(x+2)(x+3)Q(x)+ax^2+bx+c}$$

6. Given that $P(x)$ is divided by x^3-x-1, express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$, and let the remainder be ax^2+bx+c.

$$\boldsymbol{P(x)=(x^3-x-1)Q(x)+ax^2+bx+c}$$

Note Summary

The general form of a remainder can be written:

① Dividing by a second-degree expression (a quadratic), the remainder must be of degree one or less. We write this as $\boldsymbol{ax+b}$.

② Dividing by a third-degree expression (a cubic), the remainder must be of degree two or less. We write this as $\boldsymbol{ax^2+bx+c}$.

J 166a KUMON

Remainder Theorem

Ex.

When $P(x)$ is divided by $x-2$, the remainder is 5. When divided by $x-3$, the remainder is 9. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$P(x)=(x-2)(x-3)Q(x)+ax+b$$

☜ First express the relationship between quotient and remainder.

$$\begin{cases} P(2)=2a+b=5 & \cdots① \\ P(3)=3a+b=9 & \cdots② \end{cases}$$

☜ From the ***Remainder Theorem***.

From ① and ②,

$a=4$, $b=-3$

Substituting into $ax+b$.

The remainder is $4x-3$

1. When $P(x)$ is divided by $x-2$, the remainder is 3. When divided by $x+3$, the remainder is -7. Find the remainder when $P(x)$ is divided by $(x-2)(x+3)$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$P(x)=(x-2)(x+3)Q(x)+ax+b$$

$$\begin{cases} P(2)=2a+b=3 & \cdots① \\ P(-3)=-3a+b=-7 & \cdots② \end{cases}$$

From ① and ②,

$a=2$, $b=-1$

The remainder is $\mathbf{2x-1}$

2. When $P(x)$ is divided by $x-3$, the remainder is 3. When divided by $x+4$, the remainder is -4. Find the remainder when $P(x)$ is divided by $(x-3)(x+4)$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$P(x)=(x-3)(x+4)Q(x)+ax+b$$

$$\begin{cases} P(3)=3a+b=3 & \cdots① \\ P(-4)=-4a+b=-4 & \cdots② \end{cases}$$

From ① and ②,

$a=1$, $b=0$

The remainder is $\boldsymbol{x}$

3. $P(x)$ is divisible by $x-4$. When divided by $x+3$, the remainder is 14. Find the remainder when $P(x)$ is divided by x^2-x-12. *Hint*

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$\begin{aligned} P(x) &= (x^2-x-12)Q(x)+ax+b \\ &= (x-4)(x+3)Q(x)+ax+b \end{aligned}$$

$$\begin{cases} P(4)=4a+b=0 & \cdots① \\ P(-3)=-3a+b=14 & \cdots② \end{cases}$$

From ① and ②,

$a=-2$, $b=8$

The remainder is $\boldsymbol{-2x+8}$

Hint

Use the factorised form of x^2-x-12.

J 167a KUMON

Remainder Theorem

1. When $P(x)$ is divided by $x+1$, $x+2$ and $x+3$, the remainders are 2, 3 and 6 respectively. Find the remainder when $P(x)$ is divided by $(x+1)(x+2)(x+3)$.
(Remember that, when the divisor is a third-degree expression, the remainder is written as ax^2+bx+c.)

[Sol] Let the quotient be $Q(x)$, and let the remainder be ax^2+bx+c.

$$P(x)=(x+1)(x+2)(x+3)Q(x)+ax^2+bx+c$$

$$\begin{cases} P(-1)=a-b+c=2 & \cdots① \\ P(-2)=4a-2b+c=3 & \cdots② \\ P(-3)=9a-3b+c=6 & \cdots③ \end{cases}$$

From ③ − ①,

$$8a-2b=4 \qquad \cdots④$$

From ② − ①,

$$3a-b=1$$

Therefore, $b=3a-1$ $\cdots⑤$

Substituting ⑤ into ④,

$$8a-2(3a-1)=4$$
$$a=1$$

Substituting into ⑤,

$$b=2$$

Substituting into ①,

$$c=3$$

The remainder is $\boldsymbol{x^2+2x+3}$

J167b

2. When $P(x)$ is divided by $x-1$, the remainder is 1. When divided by $(x-2)(x-3)$, the remainder is 5. Find the remainder when $P(x)$ is divided by $(x-1)(x-2)(x-3)$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be ax^2+bx+c.

$$P(x)=(x-1)(x-2)(x-3)Q(x)+ax^2+bx+c \quad \cdots ⓐ$$

$$\begin{cases} P(1)=a+b+c=1 & \cdots ① \\ P(2)=4a+2b+c=5 & \cdots ② \\ P(3)=9a+3b+c=5 & \cdots ③ \end{cases}$$

From ③ $-$ ①,

$$8a+2b=4 \quad \cdots ④$$

From ② $-$ ①,

$$3a+b=4$$

Therefore, $b=-3a+4 \quad \cdots ⑤$

Substituting ⑤ into ④,

$$8a+2(-3a+4)=4$$
$$a=-2$$

Substituting into ⑤,

$$b=10$$

Substituting into ①,

$$c=-7$$

The remainder is $\mathbf{-2x^2+10x-7}$

Note: When $P(x)$ is divided by $(x-2)(x-3)$, the remainder is 5, so $P(x)$ can be written in the form

$$P(x)=(x-2)(x-3)Q_1(x)+5$$

Substituting $x=2$ into this equation gives $P(2)=5$. Also, substituting $x=2$ into equation ⓐ above gives $P(2)=4a+2b+c$. Therefore we can conclude that $P(2)=4a+2b+c=5$.

J 168a KUMON

Remainder Theorem

1. Find the remainder when $P(x)=x^{10}-3$ is divided by x^2-1.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$\begin{aligned} P(x)=x^{10}-3 &= (x^2-1)Q(x)+ax+b \\ &= (x+1)(x-1)Q(x)+ax+b \end{aligned}$$

$$\begin{cases} P(1)=1^{10}-3=-2=a+b & \cdots① \\ P(-1)=(-1)^{10}-3=-2=-a+b & \cdots② \end{cases}$$

From ① and ②,

$a=0,\ b=-2$

The remainder is $\mathbf{-2}$

J 168b

2. Find the remainder when $P(x)=32x^5-1$ is divided by $4x^2-1$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$\begin{aligned}P(x)=32x^5-1&=(4x^2-1)Q(x)+ax+b\\&=(2x+1)(2x-1)Q(x)+ax+b\end{aligned}$$

$$\begin{cases}P\left(\dfrac{1}{2}\right)=32\times\left(\dfrac{1}{2}\right)^5-1=0=\dfrac{1}{2}a+b & \cdots①\\ P\left(-\dfrac{1}{2}\right)=32\times\left(-\dfrac{1}{2}\right)^5-1=-2=-\dfrac{1}{2}a+b & \cdots②\end{cases}$$

From ① and ②,

$a=2,\ b=-1$

The remainder is $\boldsymbol{2x-1}$

J 169a KUMON

Remainder Theorem

Ex.

Find the value of $x^4-x^3-6x^2+9x-4$ when $x=\dfrac{3+\sqrt{5}}{2}$.

[Sol] From $x=\dfrac{3+\sqrt{5}}{2}$,

$$2x-3=\sqrt{5} \qquad \cdots①$$

Squaring both sides of ①,

$$4x^2-12x+9=5$$

$$4x^2-12x+4=0$$

$$x^2-3x+1=0\cdots②$$

Let

$P(x)=x^4-x^3-6x^2+9x-4$

When $P(x)$ is divided by x^2-3x+1,

the quotient is: x^2+2x-1

and the remainder is: $\boxed{\mathbf{4x-3}}$

$$\begin{array}{r|l} & x^2+2x-1 \\ \hline x^2-3x+1 & x^4-x^3-6x^2+9x-4 \\ & x^4-3x^3+x^2 \\ \hline & 2x^3-7x^2+9x \\ & 2x^3-6x^2+2x \\ \hline & -x^2+7x-4 \\ & -x^2+3x-1 \\ \hline & \boxed{\mathbf{4x-3}} \end{array}$$

Thus, $P(x)=\underbrace{(x^2-3x+1)(x^2+2x-1)}+\boxed{\mathbf{4x-3}} \quad \cdots③$

From ② and ③,

$$P\left(\frac{3+\sqrt{5}}{2}\right)=4\times\left(\mathbf{\frac{3+\sqrt{5}}{2}}\right)-3$$

The underlined terms in ③ are 0.

$$=\boxed{\mathbf{3+2\sqrt{5}}}$$

Answers: $4x-3$, $4x-3$, $4x-3$, $\dfrac{3+\sqrt{5}}{2}$, $3+2\sqrt{5}$

J169b

1. Find the value of x^4-x^2+6x-4 when $x=\dfrac{1+\sqrt{3}\,i}{2}$.

[Sol] From $x=\dfrac{1+\sqrt{3}\,i}{2}$,

$$2x-1=\sqrt{3}\,i \qquad \cdots①$$

Squaring both sides of ①,

$$4x^2-4x+1=-3$$
$$4x^2-4x+4=0$$
$$x^2-x+1=0 \quad \cdots②$$

Let $P(x)=x^4-x^2+6x-4$

When $P(x)$ is divided by x^2-x+1,

the quotient is: x^2+x-1

and the remainder is: $4x-3$

$$\begin{array}{r|l} & x^2+x-1 \\ \hline x^2-x+1 & x^4 \qquad -x^2+6x-4 \\ & x^4-x^3+x^2 \\ \hline & \quad x^3-2x^2+6x \\ & \quad x^3-x^2+x \\ \hline & \qquad -x^2+5x-4 \\ & \qquad -x^2+x-1 \\ \hline & \qquad\qquad 4x-3 \end{array}$$

Thus, $P(x)=(x^2-x+1)(x^2+x-1)+4x-3 \quad \cdots③$

From ② and ③,

$$P\left(\frac{1+\sqrt{3}\,i}{2}\right)=4\times\left(\frac{1+\sqrt{3}\,i}{2}\right)-3=\boldsymbol{-1+2\sqrt{3}\,i}$$

J 170a KUMON

Remainder Theorem

1. Find the value of a for each of the following cases.

(1) When x^2-2x+a is divisible by $x+2$.

Let $P(x)=x^2-2x+a$

$P(-2)=4+4+a=0$

$\boldsymbol{a=-8}$

(2) When x^3-3x^2+4x+a is divisible by $x-2$.

Let $P(x)=x^3-3x^2+4x+a$

$P(2)=8-12+8+a=0$

$\boldsymbol{a=-4}$

(3) When x^3+ax^2+3x+1 is divided by $x+3$ with remainder 1.

Let $P(x)=x^3+ax^2+3x+1$

$P(-3)=-27+9a-9+1=1$

$9a=36$

$\boldsymbol{a=4}$

(4) When $3x^3-8x^2+ax-4$ is divided by $3x-2$ with remainder -2.

Let $P(x)=3x^3-8x^2+ax-4$

$$P\left(\frac{2}{3}\right)=\frac{8}{9}-\frac{32}{9}+\frac{2}{3}a-4=-2$$

$$\frac{2}{3}a=\frac{14}{3}$$

$\boldsymbol{a=7}$

J170b

2. When $P(x)$ is divided by $x-1$, the remainder is 3. When divided by $x+2$, the remainder is 6. When divided by $x-3$, the remainder is 11. Find the remainder when $P(x)$ is divided by $(x-1)(x+2)(x-3)$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be ax^2+bx+c.

$$P(x)=(x-1)(x+2)(x-3)Q(x)+ax^2+bx+c$$

$$\begin{cases} P(1)=a+b+c=3 & \cdots ① \\ P(-2)=4a-2b+c=6 & \cdots ② \\ P(3)=9a+3b+c=11 & \cdots ③ \end{cases}$$

From ③−①,

$$8a+2b=8 \qquad \cdots ④$$

From ②−①,

$$3a-3b=3$$
$$a-b=1$$

Therefore, $b=a-1$ $\cdots ⑤$

Substituting ⑤ into ④,

$$8a+2(a-1)=8$$
$$a=1$$

Substituting into ⑤,

$$b=0$$

Substituting into ①,

$$c=2$$

The remainder is $\boldsymbol{x^2+2}$

J 171 a KUMON

Factor Theorem

Solve the following equations.

Ex.

$x^3-x^2+x-1=0$

[Sol] $x^2(x-1)+(x-1)=0$

$(x-1)(x^2+1)=0$ $x-1=0$ or $x^2+1=0$

$x=1$ or $x^2=-1$ If $x^2=-1$, then $x=\pm i$.

Thus, $x=1,\ \pm i$

(1) $x^3-2x^2+x-2=0$

$x^2(x-2)+(x-2)=0$
$(x-2)(x^2+1)=0$
$\boldsymbol{x=2,\ \pm i}$

(2) $2x^3-5x^2+5x-2=0$

Use the formulas
$a^3+b^3=(a+b)(a^2-ab+b^2)$
$a^3-b^3=(a-b)(a^2+ab+b^2)$
[J 47a]

$2(x^3-1)-5x(x-1)=0$
$2(x-1)(x^2+x+1)-5x(x-1)=0$
$(x-1)(2x^2-3x+2)=0$
$\boldsymbol{x=1,\ \dfrac{3\pm\sqrt{7}i}{4}}$

J171b

(3) $x^3+1=0$

$(x+1)(x^2-x+1)=0$

$\boldsymbol{x=-1,\ \dfrac{1\pm\sqrt{3}i}{2}}$

Use the formulas
$a^3+b^3=(a+b)(a^2-ab+b^2)$
$a^3-b^3=(a-b)(a^2+ab+b^2)$
[J47a]

(4) $x^3-1=0$

$(x-1)(x^2+x+1)=0$

$\boldsymbol{x=1,\ \dfrac{-1\pm\sqrt{3}i}{2}}$

(5) $x^3-8=0$

$(x-2)(x^2+2x+4)=0$

$\boldsymbol{x=2,\ -1\pm\sqrt{3}i}$

(6) $x^4-1=0$

$(x^2+1)(x^2-1)=0$

$\boldsymbol{x=\pm i,\ \pm 1}$

J 172a

Factor Theorem

When $P(a)=0$, then $P(x)$ can be divided exactly by $x-a$. From this, we can conclude:

The Factor Theorem

$P(x)$ has a factor $(x-a) \iff P(a)=0$

($\iff$ means "if and only if")

Factorise $P(x)$, using the *Factor Theorem.*

Ex.

$P(x)=x^3-7x+6$

[Sol] From $P(1)=1^3-7\times1+6=0$

> Try the factors of 6: $(\pm1, \pm2, \pm3, \pm6)$ until you find one that makes $P(x)=0$.

So $P(x)$ is divisible by $x-1$.

Therefore,

$$P(x)=(x-1)(x^2+x-6)$$
$$=(x-1)(x+3)(x-2)$$

$$\begin{array}{r l}
 & x^2+x-6 \\
x-1\,) & x^3+0x^2-7x+6 \\
 & x^3-x^2 \\ \hline
 & x^2-7x \\
 & x^2-x \\ \hline
 & -6x+6 \\
 & -6x+6 \\ \hline
 & 0
\end{array}$$

(1) $P(x)=x^3-3x-2$

From $P(-1)=(-1)^3-3\times(-1)-2=0$
$P(x)$ is divisible by $(x+1)$.
Therefore,

$$\boldsymbol{P(x)}=(x+1)(x^2-x-2)$$
$$\boldsymbol{=(x+1)^2(x-2)}$$

$$\begin{array}{r l}
 & x^2-x-2 \\
x+1\,) & x^3+0x^2-3x-2 \\
 & x^3+x^2 \\ \hline
 & -x^2-3x \\
 & -x^2-x \\ \hline
 & -2x-2 \\
 & -2x-2 \\ \hline
 & 0
\end{array}$$

[Alternative Solution
$P(2)=0$
$\boldsymbol{P(x)=(x-2)(x+1)^2}$]

J172b

(2) $P(x)=x^3+2x^2-3$

From $P(1)=1^3+2\times1^2-3=0$
$P(x)$ is divisible by $(x-1)$.
Therefore,

$$\boldsymbol{P(x)=(x-1)(x^2+3x+3)}$$

$$
\begin{array}{r}
x^2+3x+3 \\
x-1\,\big)\overline{x^3+2x^2+0x-3} \\
\underline{x^3-x^2} \\
3x^2 \\
\underline{3x^2-3x} \\
3x-3 \\
\underline{3x-3} \\
0
\end{array}
$$

(3) $P(x)=x^3+6x^2+11x+6$

$P(-1)=(-1)^3+6\times(-1)^2+11\times(-1)+6=0$
$P(x)$ is divisible by $(x+1)$.
Therefore,

$$\boldsymbol{P(x)}=(x+1)(x^2+5x+6)$$
$$\boldsymbol{=(x+1)(x+3)(x+2)}$$

$$
\begin{array}{r}
x^2+5x+6 \\
x+1\,\big)\overline{x^3+6x^2+11x+6} \\
\underline{x^3+x^2} \\
5x^2+11x \\
\underline{5x^2+5x} \\
6x+6 \\
\underline{6x+6} \\
0
\end{array}
$$

(4) $P(x)=x^3+3x^2-x-3$

$P(1)=1^3+3\times1^2-1-3=0$
$P(x)$ is divisible by $(x-1)$.
Therefore,

$$\boldsymbol{P(x)}=(x-1)(x^2+4x+3)$$
$$\boldsymbol{=(x-1)(x+3)(x+1)}$$

$$
\begin{array}{r}
x^2+4x+3 \\
x-1\,\big)\overline{x^3+3x^2-x-3} \\
\underline{x^3-x^2} \\
4x^2-x \\
\underline{4x^2-4x} \\
3x-3 \\
\underline{3x-3} \\
0
\end{array}
$$

J 173a KUMON

Factor Theorem

Factorise $P(x)$, using the *Factor Theorem.*

Ex.

$P(x)=x^3+3x^2-71x-10$

[Sol] $P(-10)=(-10)^3+3\times(-10)^2-71\times(-10)-10$ ☜ This is a factor of the constant term -10.

$\qquad\qquad = 0$

Therefore,

$$P(x)=(x+10)(x^2-7x-1)$$

▶ It is unnecessary to substitute numbers other than the factors of 10.

$$\begin{array}{r}
x^2-7x-1 \\
x+10\,\overline{)\,x^3+3x^2-71x-10} \\
\underline{x^3+10x^2} \\
-7x^2-71x \\
\underline{-7x^2-70x} \\
-x-10 \\
\underline{-x-10} \\
0
\end{array}$$

Note: If $P(x)=x^3+3x^2-71x-10$ has a factor $x-a$ (where a is an integer), then a is a factor of the constant term, -10. Try $\pm1, \pm2, \pm5, \pm10$.

(1) $P(x)=x^3-4x^2-4x-5$

$P(5)=5^3-4\times5^2-4\times5-5=0$

Therefore,

$$\boldsymbol{P(x)=(x-5)(x^2+x+1)}$$

$$\begin{array}{r}
x^2+x+1 \\
x-5\,\overline{)\,x^3-4x^2-4x-5} \\
\underline{x^3-5x^2} \\
x^2-4x \\
\underline{x^2-5x} \\
x-5 \\
\underline{x-5} \\
0
\end{array}$$

(2) $P(x)=x^3-7x^2+7x-6$

$P(6)=6^3-7\times6^2+7\times6-6=0$

Therefore,

$$\boldsymbol{P(x)=(x-6)(x^2-x+1)}$$

$$\begin{array}{r}
x^2-x+1 \\
x-6\,\overline{)\,x^3-7x^2+7x-6} \\
\underline{x^3-6x^2} \\
-x^2+7x \\
\underline{-x^2+6x} \\
x-6 \\
\underline{x-6} \\
0
\end{array}$$

J173b

(3) $P(x)=x^3+4x^2-8$

$P(-2)=(-2)^3+4\times(-2)^2-8=0$
Therefore,
$P(x)=(x+2)(x^2+2x-4)$

$$
\begin{array}{r@{}l}
 & x^2+2x-4 \\
x+2 \,\big) & \overline{x^3+4x^2+0x-8} \\
 & \underline{x^3+2x^2} \\
 & 2x^2 \\
 & \underline{2x^2+4x} \\
 & -4x-8 \\
 & \underline{-4x-8} \\
 & 0
\end{array}
$$

(4) $P(x)=x^3-4x+15$

$P(-3)=(-3)^3-4\times(-3)+15=0$
Therefore,
$P(x)=(x+3)(x^2-3x+5)$

$$
\begin{array}{r@{}l}
 & x^2-3x+5 \\
x+3 \,\big) & \overline{x^3+0x^2-4x+15} \\
 & \underline{x^3+3x^2} \\
 & -3x^2-4x \\
 & \underline{-3x^2-9x} \\
 & 5x+15 \\
 & \underline{5x+15} \\
 & 0
\end{array}
$$

J 174a KUMON

Factor Theorem

Factorise by applying the *Factor Theorem.*

Ex.

$P(x)=2x^3-3x^2-x-10$ ☜ This may be factorised as $(2x-\bigcirc)(x^2+\triangle x+\square)$.

[Sol] $P\left(\frac{5}{2}\right)=2\times\left(\frac{5}{2}\right)^3-3\times\left(\frac{5}{2}\right)^2-\frac{5}{2}-10$ ☜ Try $\pm\frac{1}{2}$, $\pm\frac{5}{2}$.

$=0$

Therefore,

$P(x)=(2x-5)(x^2+x+2)$

$$\begin{array}{r|l} & x^2+x\quad+2 \\ 2x-5 & 2x^3-3x^2-\ x-10 \\ & \underline{2x^3-5x^2} \\ & \qquad 2x^2-\ x \\ & \qquad \underline{2x^2-5x} \\ & \qquad\qquad 4x-10 \\ & \qquad\qquad \underline{4x-10} \\ & \qquad\qquad\qquad 0 \end{array}$$

Note: The numerator of the fraction substituted is the '$\bigcirc$' in $(2x-\bigcirc)(x^2+\triangle x+\square)$, and must be a factor of 10.

(1) $P(x)=2x^3+x^2+x-1$

$P\left(\frac{1}{2}\right)=2\times\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^2+\frac{1}{2}-1=0$

Therefore,

$\boldsymbol{P(x)=(2x-1)(x^2+x+1)}$

$$\begin{array}{r|l} & x^2+x\ +1 \\ 2x-1 & 2x^3+x^2+\ x-1 \\ & \underline{2x^3-x^2} \\ & \qquad 2x^2+\ x \\ & \qquad \underline{2x^2-\ x} \\ & \qquad\qquad 2x-1 \\ & \qquad\qquad \underline{2x-1} \\ & \qquad\qquad\quad 0 \end{array}$$

(2) $P(x)=2x^3-x^2+9$

$P\left(-\frac{3}{2}\right)=2\times\left(-\frac{3}{2}\right)^3-\left(-\frac{3}{2}\right)^2+9=0$

Therefore,

$\boldsymbol{P(x)=(2x+3)(x^2-2x+3)}$

$$\begin{array}{r|l} & x^2-2x\ +3 \\ 2x+3 & 2x^3-\ x^2+0x+9 \\ & \underline{2x^3+3x^2} \\ & \qquad -4x^2 \\ & \qquad \underline{-4x^2-6x} \\ & \qquad\qquad 6x+9 \\ & \qquad\qquad \underline{6x+9} \\ & \qquad\qquad\quad 0 \end{array}$$

J174b

(3) $P(x)=2x^3+3x^2+2x-2$

$$P\left(\frac{1}{2}\right)=2\times\left(\frac{1}{2}\right)^3+3\times\left(\frac{1}{2}\right)^2+2\times\frac{1}{2}-2=0$$

Therefore,

$$\boldsymbol{P(x)=(2x-1)(x^2+2x+2)}$$

$$\begin{array}{r}
x^2+2x\ +2 \\
2x-1\,\overline{)\,2x^3+3x^2+2x-2} \\
\underline{2x^3-\ \ x^2} \\
4x^2+2x \\
\underline{4x^2-2x} \\
4x-2 \\
\underline{4x-2} \\
0
\end{array}$$

This coefficient is 3, so try $\pm\frac{1}{3}, \pm\frac{2}{3}$.

(4) $P(x)=3x^3-4x^2-7x-2$

$$P\left(-\frac{2}{3}\right)=3\times\left(-\frac{2}{3}\right)^3-4\times\left(-\frac{2}{3}\right)^2-7\times\left(-\frac{2}{3}\right)-2=0$$

Therefore,

$$\boldsymbol{P(x)=(3x+2)(x^2-2x-1)}$$

$$\begin{array}{r}
x^2-2x\ -1 \\
3x+2\,\overline{)\,3x^3-4x^2-7x-2} \\
\underline{3x^3+2x^2} \\
-6x^2-7x \\
\underline{-6x^2-4x} \\
-3x-2 \\
\underline{-3x-2} \\
0
\end{array}$$

J 175a KUMON

Factor Theorem

Solve the following equations.

Ex.

$3x^3+7x^2-4=0$

[Sol] Let $P(x)=3x^3+7x^2-4$,

$P(-1)=0$, so

$P(x)$ is divisible by $x+1$.

$P(x)=(x+1)(3x^2+4x-4)$

$=(x+1)(3x-2)(x+2)$

Solving $(x+1)(3x-2)(x+2)=0$,

$x=-1,\ \frac{2}{3},\ -2$

(1) $3x^3-2x^2-12x+8=0$

Let $P(x)=3x^3-2x^2-12x+8$

$P(2)=0$, so

$P(x)$ is divisible by $x-2$.

$P(x)=(x-2)(3x^2+4x-4)$

$=(x-2)(3x-2)(x+2)$

$$\begin{array}{r|l} & 3x^2+4x-4 \\ \hline x-2 & 3x^3-2x^2-12x+8 \\ & 3x^3-6x^2 \\ \hline & 4x^2-12x \\ & 4x^2-8x \\ \hline & -4x+8 \\ & -4x+8 \\ \hline & 0 \end{array}$$

Solving $(x-2)(3x-2)(x+2)=0$,

$\boldsymbol{x=2,\ \frac{2}{3},\ -2}$

J175b

(2) $12x^3+4x^2-3x-1=0$

Let $P(x)=12x^3+4x^2-3x-1$

$P\left(\frac{1}{2}\right)=0$, so

$P(x)$ is divisible by $2x-1$.

$$P(x)=(2x-1)(6x^2+5x+1)$$
$$=(2x-1)(3x+1)(2x+1)$$

$$\begin{array}{r|l} & 6x^2+5x+1 \\ \hline 2x-1 & 12x^3+4x^2-3x-1 \\ & \underline{12x^3-6x^2} \\ & \quad 10x^2-3x \\ & \quad \underline{10x^2-5x} \\ & \qquad 2x-1 \\ & \qquad \underline{2x-1} \\ & \qquad\quad 0 \end{array}$$

Solving $(2x-1)(3x+1)(2x+1)=0$,

$$\boldsymbol{x=\frac{1}{2},\ -\frac{1}{3},\ -\frac{1}{2}}$$

(3) $4x^3-4x^2-7x+6=0$

Let $P(x)=4x^3-4x^2-7x+6$

$P\left(\frac{3}{2}\right)=0$, so

$P(x)$ is divisible by $2x-3$.

$$P(x)=(2x-3)(2x^2+x-2)$$

$$\begin{array}{r|l} & 2x^2+x-2 \\ \hline 2x-3 & 4x^3-4x^2-7x+6 \\ & \underline{4x^3-6x^2} \\ & \quad 2x^2-7x \\ & \quad \underline{2x^2-3x} \\ & \qquad -4x+6 \\ & \qquad \underline{-4x+6} \\ & \qquad\quad 0 \end{array}$$

Solving $(2x-3)(2x^2+x-2)=0$,

$$\boldsymbol{x=\frac{3}{2},\ \frac{-1\pm\sqrt{17}}{4}}$$

J 176a KUMON

Factor Theorem

Solve the following equations.

(1) $3x^3+5x^2-4x-4=0$

Let $P(x)=3x^3+5x^2-4x-4$

$P(1)=0$, so

$$P(x)=(x-1)(3x^2+8x+4)$$
$$=(x-1)(3x+2)(x+2)$$

Solving $(x-1)(3x+2)(x+2)=0$,

$$\boldsymbol{x=1,\ -\frac{2}{3},\ -2}$$

$$\begin{array}{r|l}
 & 3x^2+8x+4 \\ \hline
x-1 & 3x^3+5x^2-4x-4 \\
 & \underline{3x^3-3x^2} \\
 & 8x^2-4x \\
 & \underline{8x^2-8x} \\
 & 4x-4 \\
 & \underline{4x-4} \\
 & 0
\end{array}$$

(2) $2x^3-x^2-15x+18=0$

Let $P(x)=2x^3-x^2-15x+18$

$P(2)=0$, so

$$P(x)=(x-2)(2x^2+3x-9)$$
$$=(x-2)(2x-3)(x+3)$$

Solving $(x-2)(2x-3)(x+3)=0$,

$$\boldsymbol{x=2,\ \frac{3}{2},\ -3}$$

$$\begin{array}{r|l}
 & 2x^2+3x-9 \\ \hline
x-2 & 2x^3-x^2-15x+18 \\
 & \underline{2x^3-4x^2} \\
 & 3x^2-15x \\
 & \underline{3x^2-6x} \\
 & -9x+18 \\
 & \underline{-9x+18} \\
 & 0
\end{array}$$

Ex.

$x^4+x^3-2x^2-x+1=0$

[Sol] Let $P(x)=x^4+x^3-2x^2-x+1$

$P(1)=0$, so

$P(x)=(x-1)(x^3+2x^2-1)$ $\cdots$①

Let $Q(x)=x^3+2x^2-1$

Factorise x^3+2x^2-1 using the ***Factor Theorem***.

$Q(-1)=0$, so

$Q(x)=(x+1)(x^2+x-1)$ $\cdots$②

From ① and ②,

$P(x)=(x-1)Q(x)$

$=(x-1)(x+1)(x^2+x-1)$

Solving $(x-1)(x+1)(x^2+x-1)=0$,

$x=1,\ -1,\ \dfrac{-1\pm\sqrt{5}}{2}$

$$
\begin{array}{r|l}
 & x^3+2x^2-1 \\
\hline
x-1 & x^4+x^3-2x^2-x+1 \\
 & x^4-x^3 \\
 & 2x^3-2x^2 \\
 & 2x^3-2x^2 \\
 & -x+1 \\
 & -x+1 \\
 & 0
\end{array}
$$

(3) $x^4+x^3-8x^2+2x+4=0$

Let $P(x)=x^4+x^3-8x^2+2x+4$

$P(1)=0$, so

$P(x)=(x-1)(x^3+2x^2-6x-4)$ $\cdots$①

Let $Q(x)=x^3+2x^2-6x-4$

$Q(2)=0$, so

$Q(x)=(x-2)(x^2+4x+2)$ $\cdots$②

From ① and ②,

$P(x)=(x-1)Q(x)$

$=(x-1)(x-2)(x^2+4x+2)$

Solving $(x-1)(x-2)(x^2+4x+2)=0$,

$\boldsymbol{x=1,\ 2,\ -2\pm\sqrt{2}}$

$$
\begin{array}{r|l}
 & x^3+2x^2-6x-4 \\
\hline
x-1 & x^4+x^3-8x^2+2x+4 \\
 & x^4-x^3 \\
 & 2x^3-8x^2 \\
 & 2x^3-2x^2 \\
 & -6x^2+2x \\
 & -6x^2+6x \\
 & -4x+4 \\
 & -4x+4 \\
 & 0
\end{array}
$$

$$
\begin{array}{r|l}
 & x^2+4x+2 \\
\hline
x-2 & x^3+2x^2-6x-4 \\
 & x^3-2x^2 \\
 & 4x^2-6x \\
 & 4x^2-8x \\
 & 2x-4 \\
 & 2x-4 \\
 & 0
\end{array}
$$

J 177a KUMON

Factor Theorem

1. Solve the following equations.

Ex.

$(x-2)(x-5)(x-7)=8\cdot5\cdot3$

[Sol] Let $P(x)=(x-2)(x-5)(x-7)-8\cdot5\cdot3$ $\cdots$①

$P(10)=8\cdot5\cdot3-8\cdot5\cdot3=0$

Therefore, $x-10$ is a factor of $P(x)$.

$P(x)=x^3-14x^2+59x-70-120$ ☜ Expanding ①.

$=x^3-14x^2+59x-190$ Find the other factor by long division.

$=(x-10)(x^2-4x+19)$

Solving $(x-10)(x^2-4x+19)=0$,

$x=10,\ 2\pm\sqrt{15}\,i$

(1) $x(x+1)(x+2)=3\cdot4\cdot5$

Let $P(x)=x(x+1)(x+2)-3\cdot4\cdot5$

$P(3)=3\cdot4\cdot5-3\cdot4\cdot5=0$

Therefore, $x-3$ is a factor of $P(x)$.

$P(x)=x^3+3x^2+2x-60$

$=(x-3)(x^2+6x+20)$

Solving $(x-3)(x^2+6x+20)=0$,

$\boldsymbol{x=3,\ -3\pm\sqrt{11}\,i}$

$$\begin{array}{r|l} & x^2+6x+20 \\ \hline x-3 & x^3+3x^2+2x-60 \\ & x^3-3x^2 \\ \hline & 6x^2+2x \\ & 6x^2-18x \\ \hline & 20x-60 \\ & 20x-60 \\ \hline & 0 \end{array}$$

J 177b

Ex.

One root of $x^3-6x^2+ax-6=0$ is 3. Find the value of a and then the other roots.

[Sol] Let $P(x)=x^3-6x^2+ax-6$

$P(3)=3a-33=0$, so Substitute $x=3$.

$a=11$

Therefore,

$P(x)=x^3-6x^2+11x-6$ Substitute $a=11$.

$=(x-3)(x^2-3x+2)$ $x-3$ is a factor of $P(x)$.

$=(x-3)(x-2)(x-1)$

Solving $(x-3)(x-2)(x-1)=0$,

$x=3,2,1$

Ans. $a=11$ and the other roots are $x=2\,,1$

2. One root of $x^3+ax^2+6x-2=0$ is 1. Find the value of a and then the other roots.

[Sol] Let $P(x)=x^3+ax^2+6x-2$

$P(1)=a+5=0$, so

$a=-5$

Therefore,

$P(x)=x^3-5x^2+6x-2$

$=(x-1)(x^2-4x+2)$

Solving $(x-1)(x^2-4x+2)=0$,

$x=1\,,2\pm\sqrt{2}$

$$
\begin{array}{r|l}
 & x^2-4x+2 \\
\hline
x-1 & x^3-5x^2+6x-2 \\
 & \underline{x^3-x^2} \\
 & -4x^2+6x \\
 & \underline{-4x^2+4x} \\
 & 2x-2 \\
 & \underline{2x-2} \\
 & 0
\end{array}
$$

Ans. $\boldsymbol{a=-5}$ and other roots are $\boldsymbol{x=2\pm\sqrt{2}}$

J 178a KUMON

Factor Theorem

1. Two of the roots of $x^4+ax^3+ax^2+11x+b=0$ are 3 and -2.

(1) Find the values of a and b.

Let $P(x)=x^4+ax^3+ax^2+11x+b$,
$P(3)=0$, so $36a+b=-114$ ⋯①
$P(-2)=0$, so $4a-b=-6$ ⋯②
From ① and ②,
$a=-3$, $b=-6$

Ans. $\boldsymbol{a=-3,\ b=-6}$

(2) Find the other roots.

Substituting a and b into the original equation,

$$\begin{aligned} P(x) &= x^4-3x^3-3x^2+11x-6 \\ &= (x-3)(x+2)(x^2-2x+1) \\ &= (x-3)(x+2)(x-1)^2 \end{aligned}$$

☜ Using the known roots, 3 and -2, $(x-3)(x+2)=x^2-x-6$

$$\begin{array}{r} x^2-2x+1 \\ x^2-x-6 \overline{\smash{\big)}\, x^4-3x^3-3x^2+11x-6} \\ \underline{x^4-x^3-6x^2} \\ -2x^3+3x^2+11x \\ \underline{-2x^3+2x^2+12x} \\ x^2-x-6 \\ \underline{x^2-x-6} \\ 0 \end{array}$$

Solving $(x-3)(x+2)(x-1)^2=0$,
$x=3,\ -2,\ 1$

Ans. The other root is $\boldsymbol{x=1}$

J 178b

2. Two of the roots of $x^4+ax^3+(a+3)x^2+16x+b=0$ are 1 and 3.
Find the values of a and b and the other roots.

[Sol] Let $P(x)=x^4+ax^3+(a+3)x^2+16x+b$

$P(1)=0$, so $2a+b=-20$ $\cdots$①

$P(3)=0$, so $36a+b=-156$ $\cdots$②

From ① and ②,

$a=-4,\ b=-12$

$$P(x)=x^4-4x^3-x^2+16x-12$$
$$=(x-1)(x-3)(x^2-4)$$
$$=(x-1)(x-3)(x+2)(x-2)$$

Using the known roots, 1 and 3,
$(x-1)(x-3)=x^2-4x+3$

Solving $(x-1)(x-3)(x+2)(x-2)=0$,

$x=1,\ 3,\ \pm 2$

$$\begin{array}{rl} & x^2 \qquad\quad -4 \\ x^2-4x+3 \,\big)\!\! & \overline{x^4-4x^3-\ \ x^2+16x-12} \\ & \underline{x^4-4x^3+3x^2} \\ & \qquad\quad -4x^2+16x-12 \\ & \qquad\quad \underline{-4x^2+16x-12} \\ & \qquad\qquad\qquad\qquad\quad 0 \end{array}$$

Ans. $\boldsymbol{a=-4}$, $\boldsymbol{b=-12}$, and the other roots are $\boldsymbol{x=\pm 2}$

J 179a KUMON

Factor Theorem

1. Solve $2x^4-3x^3-x^2-3x+2=0$ using the following method.

(1) Dividing both sides by x^2 $(x\neq 0)$ gives $2x^2-3x-1-\dfrac{3}{x}+\dfrac{2}{x^2}=0$.

Express the left hand side in terms of t, where $t=x+\dfrac{1}{x}$.

[Sol] $2x^2-3x-1-\dfrac{3}{x}+\dfrac{2}{x^2}$

$$=2\left(x^2+\frac{1}{x^2}\right)-3\left(\boldsymbol{x}+\frac{\mathbf{1}}{\boldsymbol{x}}\right)-1$$

$$=2\left[\left(x+\frac{1}{x}\right)^2-2\right]-3\left(\boldsymbol{x}+\frac{\mathbf{1}}{\boldsymbol{x}}\right)-1$$

$$=2\left(x+\frac{1}{x}\right)^2-3\left(\boldsymbol{x}+\frac{\mathbf{1}}{\boldsymbol{x}}\right)-\boxed{\mathbf{5}}$$

$$=2t^2-3t-\boxed{\mathbf{5}}$$

(2) Find the value of t first, and then x.

[Sol] $2t^2-3t-\boxed{\mathbf{5}}$

$$=(2t-\boxed{\mathbf{5}})(t+\boxed{\mathbf{1}})=0$$

$$t=\frac{\boxed{\mathbf{5}}}{2},\ \boxed{\mathbf{-1}}$$

(i) If $x+\dfrac{1}{x}=\dfrac{\boxed{\mathbf{5}}}{2}$

Multiply both sides by $2x$,

$$2x^2-5x+2=0$$

$$(2x-1)(x-2)=0$$

$$x=\frac{1}{2},\ 2$$

(ii) If $x+\dfrac{1}{x}=\boxed{\mathbf{-1}}$

Multiply both sides by x,

$$x^2+x+1=0$$

$$x=\frac{-1\pm\sqrt{3}\,i}{2}$$

Ans. $\boldsymbol{x}=\dfrac{\mathbf{1}}{\mathbf{2}},\ \mathbf{2},\ \dfrac{\mathbf{-1\pm\sqrt{3}}\boldsymbol{i}}{\mathbf{2}}$

J179b

2. Solve the equation $2x^4-x^3-6x^2-x+2=0$. $\left(\text{Hint: Let } x+\dfrac{1}{x}=t\right)$

[Sol] Divide both sides by x^2 $(x\neq 0)$,

$$2x^2-x-6-\frac{1}{x}+\frac{2}{x^2}=0$$

Express the left hand side in terms of t,

$$\begin{aligned}
&2x^2-x-6-\frac{1}{x}+\frac{2}{x^2}\\
&=2\left(x+\frac{1}{x}\right)^2-\left(x+\frac{1}{x}\right)-10\\
&=2t^2-t-10
\end{aligned}$$

Find the value of t first,

$$\begin{aligned}
&2t^2-t-10\\
&=(2t-5)(t+2)=0\\
&t=\frac{5}{2},\ -2
\end{aligned}$$

(i) If $x+\dfrac{1}{x}=\dfrac{5}{2}$

Multiply both sides by $2x$,

$$\begin{aligned}
&2x^2-5x+2=0\\
&(2x-1)(x-2)=0\\
&x=\frac{1}{2},\ 2
\end{aligned}$$

(ii) If $x+\dfrac{1}{x}=-2$

Multiply both sides by x,

$$\begin{aligned}
&x^2+2x+1=0\\
&(x+1)^2=0\\
&x=-1
\end{aligned}$$

Ans. $\boldsymbol{x=\frac{1}{2},\ 2,\ -1}$

J 180a KUMON

Factor Theorem

1. Factorise using the *Factor Theorem*.

(1) $P(x)=x^3+6x^2+11x+6$

$P(-1)=0$, so

$\boldsymbol{P(x)}=(x+1)(x^2+5x+6)$

$\boldsymbol{=(x+1)(x+3)(x+2)}$

$$
\begin{array}{r|l}
 & x^2+5x+6 \\
\hline
x+1 & x^3+6x^2+11x+6 \\
 & \underline{x^3+\ x^2} \\
 & \quad 5x^2+11x \\
 & \quad \underline{5x^2+\ 5x} \\
 & \qquad 6x+6 \\
 & \qquad \underline{6x+6} \\
 & \qquad\quad 0
\end{array}
$$

(2) $P(x)=2x^3+3x^2+2x-2$

$P\left(\frac{1}{2}\right)=0$, so

$\boldsymbol{P(x)=(2x-1)(x^2+2x+2)}$

$$
\begin{array}{r|l}
 & x^2+2x+2 \\
\hline
2x-1 & 2x^3+3x^2+2x-2 \\
 & \underline{2x^3-\ x^2} \\
 & \quad 4x^2+2x \\
 & \quad \underline{4x^2-2x} \\
 & \qquad 4x-2 \\
 & \qquad \underline{4x-2} \\
 & \qquad\quad 0
\end{array}
$$

J180b

2. Solve the following equations.

(1) $x^3-2x^2+3x-6=0$

Let $P(x)=x^3-2x^2+3x-6$

$P(2)=0$, so

$\quad P(x)=(x-2)(x^2+3)$

Solving $(x-2)(x^2+3)=0$,

$\quad \boldsymbol{x=2,\ \pm\sqrt{3}i}$

$$
\begin{array}{r}
x^2+3 \\
x-2\,\overline{)\,x^3-2x^2+3x-6} \\
\underline{x^3-2x^2} \\
3x-6 \\
\underline{3x-6} \\
0
\end{array}
$$

(2) $x^4-3x^3+3x^2+x-6=0$

Let $P(x)=x^4-3x^3+3x^2+x-6$

$P(-1)=0$, so

$\quad P(x)=(x+1)(x^3-4x^2+7x-6)\ \cdots①$

Let $Q(x)=x^3-4x^2+7x-6$

$Q(2)=0$, so

$\quad Q(x)=(x-2)(x^2-2x+3)\qquad\cdots②$

From ① and ②,

$\quad P(x)=(x+1)Q(x)$

$\quad\qquad\ =(x+1)(x-2)(x^2-2x+3)$

Solving $(x+1)(x-2)(x^2-2x+3)=0$,

$\quad \boldsymbol{x=-1,\ 2,\ 1\pm\sqrt{2}i}$

$$
\begin{array}{r}
x^3-4x^2+7x-6 \\
x+1\,\overline{)\,x^4-3x^3+3x^2+x-6} \\
\underline{x^4+x^3} \\
-4x^3+3x^2 \\
\underline{-4x^3-4x^2} \\
7x^2+x \\
\underline{7x^2+7x} \\
-6x-6 \\
\underline{-6x-6} \\
0
\end{array}
$$

$$
\begin{array}{r}
x^2-2x+3 \\
x-2\,\overline{)\,x^3-4x^2+7x-6} \\
\underline{x^3-2x^2} \\
-2x^2+7x \\
\underline{-2x^2+4x} \\
3x-6 \\
\underline{3x-6} \\
0
\end{array}
$$

J181a KUMON

Proof of Identities

Find the values of a, b and c so that the following equations are true for any value of x.

Ex.

$(ax+b)(x+1)=3x^2+5x+2$ ☜ Expand the left hand side, and arrange in powers of x.

[Sol] $ax^2+(a+b)x+b=3x^2+5x+2$

Comparing the coefficients of the left and right hand sides,

$$\begin{cases} a=3 & \cdots① \\ a+b=5 & \cdots② \\ b=2 & \cdots③ \end{cases}$$

☜ From the coefficients of x^2.
☜ From the coefficients of x.
☜ From the constants.

From ①, ② and ③,

$a=3,\ b=2$

(1) $(ax+2b)(x-1)=2x^2-x-1$

$ax^2+(-a+2b)x-2b=2x^2-x-1$

Comparing the coefficients of the left and right hand sides,

$$\begin{cases} a=2 & \cdots① \\ -a+2b=-1 & \cdots② \\ -2b=-1 & \cdots③ \end{cases}$$

From ①, ② and ③,

$\boldsymbol{a=2,\ b=\frac{1}{2}}$

J181b

Use the formula $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$ [J7a]

(2) $x^3-6x^2+15x-7=(x+a)^3+bx+c$

$$x^3-6x^2+15x-7 = x^3+3x^2a+3xa^2+a^3+bx+c$$
$$= x^3+3ax^2+(3a^2+b)x+(a^3+c)$$

Comparing the coefficients of the left and right hand sides,

$$\begin{cases} 3a=-6 & \cdots① \\ 3a^2+b=15 & \cdots② \\ a^3+c=-7 & \cdots③ \end{cases}$$

From ①, $a=-2$

Substituting $a=-2$ into ②,

$3(-2)^2+b=15$

$12+b=15$

$b=3$

Substituting $a=-2$ into ③,

$(-2)^3+c=-7$

$-8+c=-7$

$c=1$

Therefore, $\boldsymbol{a=-2,\ b=3,\ c=1}$

Note:

① From the example, when $a=3$ and $b=2$,

$(3x+2)(x+1)=3x^2+5x+2$

We can substitute any value of x into both sides, e.g. taking $x=4$,

LHS $=(3\cdot 4+2)(4+1)=70$

RHS $=3\cdot 4^2+5\cdot 4+2=70$

We find LHS = RHS.

In fact this equation is true for any value of x: we call this an ***identity***.

② The equation $ax^2+bx+c=a'x^2+b'x+c'$ becomes an ***identity*** if and only if $a=a',\ b=b',\ c=c'$.

(Similar conditions apply for a cubic equation.)

J 182a KUMON

Proof of Identities

Find the values of a, b and c so that the following equations become identities.

Ex.

$3x^2+4x+a=b(x+c)^2$ ☜ Expand the RHS and arrange in powers of x.

[Sol] $3x^2+4x+a=bx^2+2bcx+bc^2$

$$\begin{cases} 3=b & \cdots① \\ 4=2bc & \cdots② \\ a=bc^2 & \cdots③ \end{cases}$$

From ①, ② and ③,

$$a=\frac{4}{3},\ b=3,\ c=\frac{2}{3}$$

(1) $x^3+ax+2=(x+1)(x^2+bx+c)$

$x^3+ax+2=x^3+(b+1)x^2+(b+c)x+c$

$$\begin{cases} 0=b+1 & \cdots① \\ a=b+c & \cdots② \\ 2=c & \cdots③ \end{cases}$$

There is no x^2 term on the LHS.

From ①, ② and ③,

$\boldsymbol{a=1,\ b=-1,\ c=2}$

Note: The method shown in the examples of J 181a and J 182a is called the ***Coefficient Comparison Method***.

Ex.

$x^2 = a(x-1)(x-2)+b(x-1)+c$

[Sol] When $x=1$, $\quad 1=c \quad \cdots①$ ☜ Substitute $x=1$ in both sides.

When $x=2$, $\quad 4=b+c \quad \cdots②$ ☜ Substitute $x=2$ in both sides.

When $x=3$, $\quad 9=2a+2b+c \cdots③$ ☜ Substitute $x=3$ in both sides.

From ①, ② and ③,

$a=1,\ b=3,\ c=1$

(This method is called the ***Value Substitution Method***.)

(2) $a(x-1)^2+b(x+1)^2+c=x^2$

When $x=1$, $\quad 4b+c=1 \quad \cdots①$
When $x=-1$, $\quad 4a+c=1 \quad \cdots②$
When $x=0$, $\quad a+b+c=0 \quad \cdots③$
From ①−②,

$4b-4a=0$

$a=b$

Substituting into ③,

$2a+c=0 \quad \cdots④$

From ②−④,

$2a=1$

$a=\dfrac{1}{2}$

Since $a=b$, $b=\dfrac{1}{2}$

Substituting $a=\dfrac{1}{2}$ into ④,

$2\cdot\dfrac{1}{2}+c=0$

$c=-1$

Therefore, $\boldsymbol{a=\dfrac{1}{2},\ b=\dfrac{1}{2},\ c=-1}$

(3) $ax^2+bx+c=0$ (Use the values $x=0,\ x=1,\ x=-1$)

When $x=0$, $\quad c=0 \quad \cdots①$
When $x=1$, $\quad a+b+c=0 \quad \cdots②$
When $x=-1$, $\quad a-b+c=0 \quad \cdots③$
From ①, ② and ③,

$\boldsymbol{a=0,\ b=0,\ c=0}$

Note: In (3), in order for $ax^2+bx+c=0$ to be an identity, the following must be true: $a=0$, $b=0$, $c=0$.

J183a KUMON

Proof of Identities

Determine the values of a, b, and c so that the following equations become identities. (Consider carefully whether to use the *Coefficient Comparison Method* or the *Value Substitution Method*.)

(1) $(ax+b)(x+c)=3x^2+5x+2$

Expanding the LHS,

$ax^2+(ac+b)x+bc=3x^2+5x+2$

Comparing coefficients,

$$\begin{cases} a=3 & \cdots ① \\ ac+b=5 & \cdots ② \\ bc=2 & \cdots ③ \end{cases}$$

Substituting ① into ②,

$3c+b=5$

$b=5-3c \quad \cdots ④$

Substituting ④ into ③,

$(5-3c)\cdot c=2$

$3c^2-5c+2=0$

$(3c-2)(c-1)=0$

$c=\dfrac{2}{3},\ 1$

(i) When $c=\dfrac{2}{3}$,

$b=5-3\times\dfrac{2}{3}=3$

(ii) When $c=1$,

$b=5-3=2$

Therefore, $\boldsymbol{a=3,\ b=3,\ c=\dfrac{2}{3}}$, or $\boldsymbol{a=3,\ b=2,\ c=1}$

J183b

(2) $ax^3 - x^2 - 4x + 5 = (2x-1)(bx^2 + cx - 5)$

Expanding the right hand side,

$$ax^3 - x^2 - 4x + 5 = 2bx^3 + (2c-b)x^2 - (10+c)x + 5$$

Comparing coefficients,

$$\begin{cases} a = 2b & \cdots ① \\ -1 = 2c - b & \cdots ② \\ 4 = 10 + c & \cdots ③ \end{cases}$$

Solving ③ for c,

$$c = -6 \qquad \cdots ④$$

Substituting ④ into ②,

$$-12 - b = -1$$

$$b = -11 \qquad \cdots ⑤$$

Substituting ⑤ into ①,

$$a = -22$$

Therefore, $\boldsymbol{a = -22,\ b = -11,\ c = -6}$

(3) $a(x-1)(x-2) + b(x-2)(x-3) + c(x-3)(x-1) = x$

When $x = 1$, $\quad 2b = 1 \quad \cdots ①$

When $x = 2$, $\quad -c = 2 \quad \cdots ②$

When $x = 3$, $\quad 2a = 3 \quad \cdots ③$

From ①, ② and ③,

$$\boldsymbol{a = \frac{3}{2},\ b = \frac{1}{2},\ c = -2}$$

Note Summary

There are 2 ways to find the value of the coefficients in an identity.

① The ***Coefficient Comparison Method***

② The ***Value Substitution Method***

Both methods give the same results.

J 184a KUMON

Proof of Identities

Ex.

Determine the values of a and b for which x^3+ax^2+bx+1 is divisible by x^2+x+1.

[Sol] Let the quotient be $x+c$. Quotient must be a linear expression.

$$\begin{aligned} x^3+ax^2+bx+1 &= (x^2+x+1)(x+c) \\ &= x^3+(c+1)x^2+(c+1)x+c \end{aligned}$$

Expand RHS and arrange in powers of x.

Comparing coefficients,

$$\begin{cases} a=c+1 & \cdots① \\ b=c+1 & \cdots② \\ c=1 & \cdots③ \end{cases}$$

From ①, ② and ③,

$a=2$, $b=2$

1. Determine the values of a and b for which x^3+ax+b is divisible by $(x-2)^2$.

[Sol] Let the quotient be $x+c$.

$$\begin{aligned} x^3+ax+b &= (x-2)^2(x+c) \\ &= (x^2-4x+4)(x+c) \\ &= x^3+(c-4)x^2+(4-4c)x+4c \end{aligned}$$

Comparing coefficients,

$$\begin{cases} c-4=0 & \cdots① \\ 4-4c=a & \cdots② \\ 4c=b & \cdots③ \end{cases}$$

From ①, $c=4$

Substituting into ② and ③,

$\boldsymbol{a=-12}$, $\boldsymbol{b=16}$

J184b

2. Find the values of a and b so that $x^4+6x^3+7x^2+ax+b$ is a perfect square.

Hint

[Sol] Let $x^4+6x^3+7x^2+ax+b=(x^2+cx+d)^2$.

$$\begin{aligned}x^4+6x^3+7x^2+ax+b&=[(x^2+cx)+d]^2\\&=(x^2+cx)^2+2d(x^2+cx)+d^2\\&=x^4+2cx^3+(c^2+2d)x^2+2cdx+d^2\end{aligned}$$

Comparing coefficients,

$$\begin{cases}2c=6 & \cdots ① \\ c^2+2d=7 & \cdots ② \\ 2cd=a & \cdots ③ \\ d^2=b & \cdots ④\end{cases}$$

From ①, $c=3$

Substituting into ②,

$9+2d=7$

$d=-1$

Substituting into ③,

$2\times3\times(-1)=a$

$a=-6$

Substituting into ④,

$(-1)^2=b$

$b=1$

Therefore, $\boldsymbol{a=-6,\ b=1}$

Hint

Let $x^4+6x^3+7x^2+ax+b=(x^2+cx+d)^2$.

J 185a KUMON

Proof of Identities

Find the values of x and y so that the following equations are true for any value of k.

Ex.

$(k+2)x+(2k+3)y-3k+2=0$

[Sol] $(x+2y-3)k+(2x+3y+2)=0$ ☜ Arrange in powers of k.

Therefore,

$$\begin{cases} x+2y-3=0 & \cdots ① \\ 2x+3y+2=0 & \cdots ② \end{cases}$$

☜ Compare coefficients in $(x+2y-3)k+(2x+3y+2)=0\times k+0$.

From ① and ②,

$$\begin{cases} x=-13 \\ y=8 \end{cases}$$

(1) $(k+1)x-(2k+3)y-3k-5=0$

$kx+x-2ky-3y-3k-5=0$

$(x-2y-3)k+(x-3y-5)=0$

Therefore,

$$\begin{cases} x-2y-3=0 & \cdots ① \\ x-3y-5=0 & \cdots ② \end{cases}$$

From ①−②,

$y=-2$

Substituting into ①,

$x+4-3=0$

$x=-1$

Therefore,

$$\begin{cases} \boldsymbol{x=-1} \\ \boldsymbol{y=-2} \end{cases}$$

J185b

(2) $x^2+y^2+(k-2)x-2ky-k-4=0$

$x^2+y^2+kx-2x-2ky-k-4=0$

$(x-2y-1)k+(x^2+y^2-2x-4)=0$

Therefore,

$$\begin{cases} x-2y-1=0 & \cdots① \\ x^2+y^2-2x-4=0 & \cdots② \end{cases}$$

Solving ① for x,

$x=2y+1 \quad \cdots③$

Substituting into ②,

$(2y+1)^2+y^2-2(2y+1)-4=0$

$4y^2+4y+1+y^2-4y-2-4=0$

$5y^2-5=0$

$y^2=1$

$y=\pm1$

Substituting into ③,

When $y=1$, $x=3$

When $y=-1$, $x=-1$

Therefore,

$$\begin{cases} \boldsymbol{x=3} \\ \boldsymbol{y=1} \end{cases} \quad \begin{cases} \boldsymbol{x=-1} \\ \boldsymbol{y=-1} \end{cases}$$

J 186a KUMON

Proof of Identities

Ex.

Find the values of a and b so that the following equation is an identity.

$$\frac{3x-1}{(x-2)(x+3)}=\frac{a}{x-2}+\frac{b}{x+3}$$

[Sol] $\dfrac{3x-1}{(x-2)(x+3)}=\dfrac{a(x+3)+b(x-2)}{(x-2)(x+3)}$ ☜ Simplify the RHS.

$=\dfrac{(a+b)x+3a-2b}{(x-2)(x+3)}$ ☜ Arrange the numerator.

Comparing coefficients in the numerators,

$$\begin{cases} a+b=3 & \cdots ① \\ 3a-2b=-1 & \cdots ② \end{cases}$$

From ① and ②,

$a=1,\ b=2$

1. Find the values of a and b so that the following equation is an identity.

$$\frac{x-3}{x^2-3x+2}=\frac{a}{x-1}+\frac{b}{x-2}$$

Factorise the LHS, and simplify the RHS.

[Sol] $\dfrac{x-3}{(x-1)(x-2)}=\dfrac{a(x-2)+b(x-1)}{(x-1)(x-2)}$

$=\dfrac{(a+b)x-2a-b}{(x-1)(x-2)}$

Comparing coefficients in the numerators,

$$\begin{cases} a+b=1 & \cdots ① \\ -2a-b=-3 & \cdots ② \end{cases}$$

From ①, $b=-a+1 \quad \cdots ③$

Substituting ③ into ②,

$-2a-(-a+1)=-3$

$-a-1=-3$

$a=2$

Substituting into ③,

$b=-1$

Therefore, $\boldsymbol{a=2,\ b=-1}$

J186b

2. Find the values of a, b and c so that the following equation is true for any value of x.

$$\frac{4x-2}{x^3-3x^2+2x}=\frac{a}{x}+\frac{b}{x-1}+\frac{c}{x-2}$$

Factorise the LHS.

[Sol] LHS $=\dfrac{4x-2}{x(x^2-3x+2)}=\dfrac{4x-2}{x(x-1)(x-2)}$

Simplify the RHS.

$$\begin{aligned}\text{RHS}&=\frac{a(x-1)(x-2)+bx(x-2)+cx(x-1)}{x(x-1)(x-2)}\\&=\frac{ax^2-3ax+2a+bx^2-2bx+cx^2-cx}{x(x-1)(x-2)}\\&=\frac{(a+b+c)x^2-(3a+2b+c)x+2a}{x(x-1)(x-2)}\end{aligned}$$

Comparing coefficients in the numerators,

$$\begin{cases}a+b+c=0 & \cdots(1)\\-(3a+2b+c)=4 & \cdots(2)\\2a=-2 & \cdots(3)\end{cases}$$

From (3), $a=-1$

Substituting into (1) and (2),

$$-1+b+c=0 \quad \cdots(4)$$
$$-(-3+2b+c)=4 \quad \cdots(5)$$

From (4), $b=-c+1 \quad \cdots(6)$

Substituting (6) into (5),

$$-[-3+2(-c+1)+c]=4$$
$$-(-c-1)=4$$
$$c+1=4$$
$$c=3$$

Substituting into (6),

$$b=-2$$

Therefore, $\boldsymbol{a=-1,\ b=-2,\ c=3}$

J 187a KUMON

Proof of Identities

Prove the following identities.

Ex.

$(x+y)^2+(x-y)^2=2(x^2+y^2)$

[Sol] LHS $= x^2+2xy+y^2+x^2-2xy+y^2$

$= 2x^2+2y^2$

RHS $= 2x^2+2y^2$

Thus, LHS = RHS

(1) $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$

LHS $= a^2c^2+a^2d^2+b^2c^2+b^2d^2$

RHS $= a^2c^2+2abcd+b^2d^2+a^2d^2-2abcd+b^2c^2$
$= a^2c^2+a^2d^2+b^2c^2+b^2d^2$

Thus, LHS = RHS

J187b

(2) $(x^2-1)(y^2-1)=(xy+1)^2-(x+y)^2$

$$\begin{aligned}\text{LHS}&=x^2y^2-x^2-y^2+1\end{aligned}$$

$$\begin{aligned}\text{RHS}&=(x^2y^2+2xy+1)-(x^2+2xy+y^2)\\&=x^2y^2-x^2-y^2+1\end{aligned}$$

Thus, LHS = RHS

(3) $(a+b+c)(ab+bc+ca)=(a+b)(b+c)(c+a)+abc$

$$\begin{aligned}\text{LHS}&=a^2b+abc+a^2c+ab^2+b^2c+abc+abc+bc^2+ac^2\\&=a^2b+a^2c+ab^2+b^2c+bc^2+ac^2+3abc\end{aligned}$$

$$\begin{aligned}\text{RHS}&=(ab+ac+b^2+bc)(c+a)+abc\\&=abc+a^2b+ac^2+a^2c+b^2c+ab^2+bc^2+abc+abc\\&=a^2b+a^2c+ab^2+b^2c+bc^2+ac^2+3abc\end{aligned}$$

Thus, LHS = RHS

Alternative Solution

$$\begin{aligned}\text{LHS}&=[(a+b)+c][ab+c(a+b)]\\&=ab(a+b)+c(a+b)^2+abc+c^2(a+b)\\&=(a+b)[ab+(a+b)c+c^2]+abc\\&=(a+b)(a+c)(b+c)+abc\\&=\text{RHS}\end{aligned}$$

J188a KUMON

Proof of Identities

Ex.

Given $x+y=1$, prove that $x^2+y=y^2+x$.

[Sol] If $x+y=1$, then $y=-x+1$ ···①

$\text{LHS}=x^2-x+1$ ☜ Substitute ①.

$\text{RHS}=(-x+1)^2+x$ ☜ Substitute ①.

$=x^2-x+1$

Thus, LHS = RHS

1. Given $a+b=1$, prove that $a^2+b^2+1=2(a+b-ab)$.

[Sol] If $a+b=1$, then $b=-a+1$ ···①

Substitute ①.

$\text{LHS}=a^2+(-a+1)^2+1=2a^2-2a+2$

$$\begin{aligned}\text{RHS}&=2[a+(-a+1)-a(-a+1)]\\&=2(a^2-a+1)\\&=2a^2-2a+2\end{aligned}$$

Thus, LHS = RHS

J188b

Ex.

Given $x+y=1$, prove that $x^2+y=y^2+x$.

[Sol] $(\text{LHS})-(\text{RHS}) = x^2+y-y^2-x$

$$= (x+y)(x-y)-(x-y)$$

$$= (x-y)(x+y-1)$$

$$= 0$$

$x+y=1$, so $x+y-1=0$.

Thus, LHS = RHS

2. Given $a+b+c=0$, prove that $a^2-bc=b^2-ca$.

[Sol] $(\text{LHS})-(\text{RHS}) = a^2-bc-b^2+ca$

$$= (a+b)(a-b)+c(a-b)$$

$$= (a-b)(a+b+c)$$

$$= 0$$

Thus, LHS = RHS

J 189a KUMON

Proof of Identities

Ex.

Given $a+b+c=0$, prove that $(a+b)(b+c)(c+a)+abc=0$.

[Sol] From $a+b+c=0$,

$$\begin{cases} a+b=-c & \cdots ① \\ b+c=-a & \cdots ② \\ c+a=-b & \cdots ③ \end{cases}$$

Substituting ①, ② and ③ into the LHS,

$$\begin{aligned} \text{LHS} &= (-c)(-a)(-b)+abc \\ &= -abc+abc \\ &= 0 \end{aligned}$$

Thus, LHS = RHS

1. Given $a+b+c=0$, prove that $\frac{1}{a}(b+c)+\frac{1}{b}(c+a)+\frac{1}{c}(a+b)+3=0$.

[Sol] From $a+b+c=0$,

$$\begin{cases} a+b=-c & \cdots ① \\ b+c=-a & \cdots ② \\ c+a=-b & \cdots ③ \end{cases}$$

Substituting ①, ② and ③ into the LHS,

$$\begin{aligned} \text{LHS} &= \frac{1}{a}(-a)+\frac{1}{b}(-b)+\frac{1}{c}(-c)+3 \\ &= -1-1-1+3=0 \end{aligned}$$

Thus, LHS = RHS

J189b

2. Given $a+b+c=0$, prove that $a^3+b^3+c^3=3abc$. *Hint*

[Sol] If $a+b+c=0$, then $c=-(a+b) \quad \cdots ①$

$$\begin{aligned}\text{LHS} &= a^3+b^3-(a+b)^3 \\ &= a^3+b^3-a^3-3a^2b-3ab^2-b^3 \\ &= -3a^2b-3ab^2\end{aligned}$$

Substitute ①.

Use the formula $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$$\begin{aligned}\text{RHS} &= -3ab(a+b) \\ &= -3a^2b-3ab^2\end{aligned}$$

Substitute ①.

Thus, LHS $=$ RHS

Hint

Since $a+b+c=0$, then $c=-(a+b)$.

J 190a KUMON

Proof of Identities

1. Find the values of a, b, c and d so that the following equation is an identity.

$$x^3+1=a(x-1)^3+b(x-1)^2+c(x-1)+d$$

[Sol] RHS $=ax^3+(-3a+b)x^2+(3a-2b+c)x+(-a+b-c+d)$

Comparing coefficients,

$$\begin{cases} a=1 \\ -3a+b=0 \\ 3a-2b+c=0 \\ -a+b-c+d=1 \end{cases}$$

Therefore,

$\boldsymbol{a=1,\ b=3,\ c=3,\ d=2}$

2. Given $xy=1$, prove that $\dfrac{1}{x+1}+\dfrac{1}{y+1}=1$.

[Sol] LHS $=\dfrac{y+1+x+1}{(x+1)(y+1)}=\dfrac{x+y+2}{xy+x+y+1}\quad\cdots①$

Substituting $xy=1$ into ①,

$$\text{LHS}=\frac{x+y+2}{1+x+y+1}=\frac{x+y+2}{x+y+2}=1$$

Thus, LHS $=$ RHS

Alternative Solution

$$\begin{aligned}(\text{LHS})-(\text{RHS})&=\frac{1}{x+1}+\frac{1}{y+1}-1\\ &=\frac{y+1+x+1-(x+1)(y+1)}{(x+1)(y+1)}\\ &=\frac{y+1+x+1-xy-x-y-1}{(x+1)(y+1)}\\ &=\frac{1-xy}{(x+1)(y+1)}=0\end{aligned}$$

☜ Substitute $xy=1$.

J 190b

3. Given $a+b+c=0$, prove that $a\left(\frac{1}{b}+\frac{1}{c}\right)+b\left(\frac{1}{c}+\frac{1}{a}\right)+c\left(\frac{1}{a}+\frac{1}{b}\right)+3=0$.

[Sol] LHS $= \frac{a}{b}+\frac{a}{c}+\frac{b}{c}+\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+3$

$$= \frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}+3 \quad \cdots ①$$

Since $a+b+c=0$,

$$\begin{cases} b+c=-a \\ a+c=-b \\ a+b=-c \end{cases}$$

Substituting into ①,

$$\text{LHS} = \frac{-a}{a}+\frac{-b}{b}+\frac{-c}{c}+3$$
$$= -1-1-1+3=0$$

Thus, LHS = RHS

J 191 a KUMON

Proof of Inequalities

Given the proportion $\frac{a}{b}=\frac{c}{d}$, prove the following.

Ex.

$$\frac{a-b}{a+b}=\frac{c-d}{c+d} \quad \cdots ①$$

[Sol] Let $\frac{a}{b}=\frac{c}{d}=k$, ☜ Let the given proportion equal k.

$$a=bk,\ c=dk \quad \cdots ②$$

Substituting ② into ①,

$$\text{LHS}=\frac{bk-b}{bk+b}=\frac{b(k-1)}{b(k+1)}=\frac{k-1}{k+1}$$

$$\text{RHS}=\frac{dk-d}{dk+d}=\frac{d(k-1)}{d(k+1)}=\frac{k-1}{k+1}$$

Thus, LHS = RHS

(1) $\frac{a+b}{a+2b}=\frac{c+d}{c+2d} \quad \cdots ①$

Let $\frac{a}{b}=\frac{c}{d}=k$,

$$a=bk,\ c=dk \quad \cdots ②$$

Substituting ② into ①,

$$\text{LHS}=\frac{bk+b}{bk+2b}=\frac{b(k+1)}{b(k+2)}=\frac{k+1}{k+2}$$

$$\text{RHS}=\frac{dk+d}{dk+2d}=\frac{d(k+1)}{d(k+2)}=\frac{k+1}{k+2}$$

Thus, LHS = RHS

Note: An equation that states that two ratios are equal is called a ***proportion***.
Proportions can be written as

$$\frac{a}{b}=\frac{c}{d} \quad \text{or} \quad a:b=c:d$$

Both are equivalent.

J191b

(2) $\dfrac{a(a+c)}{b(b+d)}=\dfrac{c^2}{d^2}$ $\cdots$①

Let $\dfrac{a}{b}=\dfrac{c}{d}=k$,

$a=bk,\ c=dk$ $\cdots$②

Substituting ② into ①,

$$\text{LHS}=\frac{bk(bk+dk)}{b(b+d)}=\frac{bk^2(b+d)}{b(b+d)}=k^2$$

$$\text{RHS}=\frac{(dk)^2}{d^2}=\frac{d^2k^2}{d^2}=k^2$$

Thus, LHS = RHS

(3) $\dfrac{b^2+d^2}{a^2+c^2}=\dfrac{b^2-d^2}{a^2-c^2}$ $(b^2\neq d^2)$ $\cdots$①

Let $\dfrac{a}{b}=\dfrac{c}{d}=k$,

$a=bk,\ c=dk$ $\cdots$②

Substituting ② into ①,

$$\text{LHS}=\frac{b^2+d^2}{b^2k^2+d^2k^2}=\frac{b^2+d^2}{k^2(b^2+d^2)}=\frac{1}{k^2}$$

$$\text{RHS}=\frac{b^2-d^2}{b^2k^2-d^2k^2}=\frac{b^2-d^2}{k^2(b^2-d^2)}=\frac{1}{k^2}$$

Thus, LHS = RHS

J 192a KUMON

Proof of Inequalities

Ex.

Given the proportion $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$, find the value of $\frac{x-y+z}{x+y+z}$.

[Sol] Let $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}=k$,

$$\begin{cases} x=2k & \cdots① \\ y=3k & \cdots② \\ z=4k & \cdots③ \end{cases}$$

Substituting ①, ② and ③,

$$\frac{x-y+z}{x+y+z}=\frac{2k-3k+4k}{2k+3k+4k}=\frac{3k}{9k}=\frac{1}{3}$$

1. Given the proportion $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$, find the value of $\frac{xy+yz+zx}{x^2+y^2+z^2}$.

[Sol] Let $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}=k$,

$$\begin{cases} x=3k & \cdots① \\ y=4k & \cdots② \\ z=5k & \cdots③ \end{cases}$$

Substituting ①, ② and ③,

$$\begin{aligned}\frac{xy+yz+zx}{x^2+y^2+z^2}&=\frac{3k\cdot 4k+4k\cdot 5k+5k\cdot 3k}{(3k)^2+(4k)^2+(5k)^2}\\&=\frac{47k^2}{50k^2}\\&=\mathbf{\frac{47}{50}}\end{aligned}$$

J 192b

Ex.

Given the proportion $\frac{x+2y}{4}=\frac{2x+y}{5}$, find the value of $\frac{3xy}{x^2-xy+y^2}$.

[Sol] Let $\frac{x+2y}{4}=\frac{2x+y}{5}=k$,

$$\begin{cases} x+2y=4k & \cdots ① \\ 2x+y=5k & \cdots ② \end{cases}$$

From ① and ②, $x=2k \quad \cdots ③$

$y=k \quad \cdots ④$

Substituting ③ and ④,

$$\frac{3xy}{x^2-xy+y^2}=\frac{3\cdot 2k\cdot k}{(2k)^2-2k\cdot k+k^2}=\frac{6k^2}{3k^2}=2$$

2. Given the proportion $\frac{3x+2y}{7}=\frac{2x+3y}{8}$, find the value of $\frac{xy^2-x^2y}{x^3+y^3}$.

[Sol] Let $\frac{3x+2y}{7}=\frac{2x+3y}{8}=k$,

$$\begin{cases} 3x+2y=7k & \cdots ① \\ 2x+3y=8k & \cdots ② \end{cases}$$

From ① and ②, $x=k \quad \cdots ③$

$y=2k \quad \cdots ④$

Substituting ③ and ④,

$$\frac{xy^2-x^2y}{x^3+y^3}=\frac{k\cdot(2k)^2-k^2\cdot 2k}{k^3+(2k)^3}=\frac{2k^3}{9k^3}=\mathbf{\frac{2}{9}}$$

J 193a KUMON

Proof of Inequalities

Ex.

Given $a+b+c\neq 0$, find the value of the proportion
$$\frac{b+c}{a}=\frac{c+a}{b}=\frac{a+b}{c}.$$

[Sol] Let $\dfrac{b+c}{a}=\dfrac{c+a}{b}=\dfrac{a+b}{c}=k$, ☜ Find the value of k.

$$\begin{cases} b+c=ak & \cdots① \\ c+a=bk & \cdots② \\ a+b=ck & \cdots③ \end{cases}$$

From ①+②+③,

$$2(a+b+c)=k(a+b+c)$$

$a+b+c\neq 0$ This must be true in order to divide by $(a+b+c)$.

Since $a+b+c\neq 0$, $k=2$

Thus, the proportion equals 2. $\left[\dfrac{b+c}{a}=\dfrac{c+a}{b}=\dfrac{a+b}{c}=2\right]$

1. Given $a+b+c\neq 0$, find the value of the proportion
$$\frac{c}{a+3b}=\frac{a}{b+3c}=\frac{b}{c+3a}.$$

[Sol] Let $\dfrac{c}{a+3b}=\dfrac{a}{b+3c}=\dfrac{b}{c+3a}=k$,

$$\begin{cases} c=k(a+3b) & \cdots① \\ a=k(b+3c) & \cdots② \\ b=k(c+3a) & \cdots③ \end{cases}$$

From ①+②+③,

$$\begin{aligned} a+b+c&=k(4a+4b+4c) \\ &=4k(a+b+c) \end{aligned}$$

Since $a+b+c\neq 0$, $k=\dfrac{1}{4}$

Thus, the proportion equals $\mathbf{\dfrac{1}{4}}$. $\left[\dfrac{c}{a+3b}=\dfrac{a}{b+3c}=\dfrac{b}{c+3a}=\mathbf{\dfrac{1}{4}}\right]$

J193b

2. Find the value of the proportion $\dfrac{bc+ca}{ab}=\dfrac{ca+ab}{bc}=\dfrac{ab+bc}{ca}$, in each of the following cases.

(1) When $ab+bc+ca\neq 0$

Let $\dfrac{bc+ca}{ab}=\dfrac{ca+ab}{bc}=\dfrac{ab+bc}{ca}=k$,

$$\begin{cases} bc+ca=kab & \cdots① \\ ca+ab=kbc & \cdots② \\ ab+bc=kca & \cdots③ \end{cases}$$

From ①+②+③,

$$2(ab+bc+ca)=k(ab+bc+ca)$$

Since $ab+bc+ca\neq 0$, $\quad k=2$

Thus, the proportion equals **2**.

$$\left[\dfrac{bc+ca}{ab}=\dfrac{ca+ab}{bc}=\dfrac{ab+bc}{ca}=\mathbf{2}\right]$$

(2) When $ab+bc+ca=0$

[Sol] From $ab+bc+ca=0$

$$\begin{cases} bc+ca=-ab & \cdots① \\ ca+ab=-bc & \cdots② \\ ab+bc=-ca & \cdots③ \end{cases}$$

Substituting ①, ② and ③ into $\dfrac{bc+ca}{ab}=\dfrac{ca+ab}{bc}=\dfrac{ab+bc}{ca}$,

$$\dfrac{-ab}{ab}=\dfrac{-bc}{bc}=\dfrac{-ca}{ca}=-1$$

Thus, the proportion equals **−1**.

$$\left[\dfrac{bc+ca}{ab}=\dfrac{ca+ab}{bc}=\dfrac{ab+bc}{ca}=\mathbf{-1}\right]$$

J 194a KUMON

Proof of Inequalities

Ex.

Given $x > y$, prove $4x + 2y > 2x + 4y$.

[Sol] $(\text{LHS}) - (\text{RHS}) = (4x + 2y) - (2x + 4y)$ ☜ Prove that LHS − RHS > 0.

$$= 2x - 2y$$

$$= 2(x - y)$$

As $x > y$,

$x - y > 0$,

Therefore, $(\text{LHS}) - (\text{RHS}) > 0$

Thus, $4x + 2y > 2x + 4y$ is true.

1. Given $x > y$, prove $\dfrac{x+y}{2} > \dfrac{2x+3y}{5}$.

[Sol] $(\text{LHS}) - (\text{RHS}) = \dfrac{x+y}{2} - \dfrac{2x+3y}{5}$

$$= \frac{x-y}{10}$$

As $x > y$,

$x - y > 0$

Therefore, $(\text{LHS}) - (\text{RHS}) > 0$

Thus, $\dfrac{x+y}{2} > \dfrac{2x+3y}{5}$ is true.

J194b

Ex.

Prove $x^2 > x-1$.

[Sol] $(\text{LHS})-(\text{RHS}) = x^2 - x + 1$

$$= \left(x-\frac{1}{2}\right)^2 + \frac{3}{4}$$

Complete the square.

Since $\left(x-\frac{1}{2}\right)^2 \geqslant 0$,

$$\left(x-\frac{1}{2}\right)^2 + \frac{3}{4} \geqslant \frac{3}{4} > 0$$

Therefore, $x^2 > x-1$

2. Prove $x^2 > 2x-3$.

[Sol] $(\text{LHS})-(\text{RHS}) = x^2 - 2x + 3$

$$= (x-1)^2 + 2$$

Since $(x-1)^2 \geqslant 0$,

$$(x-1)^2 + 2 \geqslant 2 > 0$$

Therefore, $x^2 > 2x-3$

J 195a KUMON

Proof of Inequalities

1. Prove the following inequalities. Then state for which values the LHS equals the RHS.

Ex.

$a^2+b^2 \geqslant 2ab$

[Sol] $(\text{LHS})-(\text{RHS}) = a^2+b^2-2ab$

$= (a-b)^2 \geqslant 0$ ☜ Changes to an equal sign when $a-b=0$.

Therefore, $a^2+b^2 \geqslant 2ab$

LHS = RHS when $a=b$

(1) $(a^2+b^2)(x^2+y^2) \geqslant (ax+by)^2$

$(\text{LHS})-(\text{RHS})$
$= (a^2+b^2)(x^2+y^2)-(ax+by)^2$
$= (a^2x^2+a^2y^2+b^2x^2+b^2y^2)-(a^2x^2+2abxy+b^2y^2)$
$= a^2y^2-2abxy+b^2x^2$
$= (ay-bx)^2 \geqslant 0$ ☜ Changes to an equal sign when $ay-bx=0$.

Thus, $(a^2+b^2)(x^2+y^2) \geqslant (ax+by)^2$

LHS = RHS when $ay-bx=0$, i.e. when $ay=bx$

J 195b

2. Prove the following inequalities. Then state for which values the LHS equals the RHS.

Ex.

$a^2+b^2 \geqslant ab$

[Sol] $(\text{LHS})-(\text{RHS}) = a^2+b^2-ab$

$$= \left(a-\frac{1}{2}b\right)^2+\frac{3}{4}b^2$$

Since $\left(a-\frac{1}{2}b\right)^2 \geqslant 0$, and $\frac{3}{4}b^2 \geqslant 0$,

$$\left(a-\frac{1}{2}b\right)^2+\frac{3}{4}b^2 \geqslant 0$$

☜ Changes to an equal sign when $a=\frac{1}{2}b$, and $\frac{3}{4}b^2=0$.

Therefore, $a^2+b^2 \geqslant ab$

LHS $=$ RHS when $a=b=0$

☜ $\frac{3}{4}b^2=0$ if and only if $b=0$
$a=\frac{1}{2}b$, so $a=0$.

(1) $a^2+b^2 \geqslant 2(a-b-1)$

$$\begin{aligned}(\text{LHS})-(\text{RHS}) &= a^2+b^2-2(a-b-1)\\ &= a^2-2a+b^2+2b+2\\ &= (a^2-2a+1)+(b^2+2b+1)\\ &= (a-1)^2+(b+1)^2\end{aligned}$$

Since $(a-1)^2 \geqslant 0$, and $(b+1)^2 \geqslant 0$,

$$(a-1)^2+(b+1)^2 \geqslant 0$$

Therefore, $a^2+b^2 \geqslant 2(a-b-1)$

LHS $=$ RHS when $a-1=0$ and $b+1=0$, i.e. when

$a=1,\ b=-1.$

J 196a KUMON

Proof of Inequalities

1. Prove the following inequalities. Then state the conditions for which the LHS equals the RHS.

Ex.

Given $a+b=1$, prove the inequality $\frac{1}{2} \geqslant 2ab$.

[Sol] $a+b=1$, so $b=1-a$

$(\text{LHS})-(\text{RHS})=\frac{1}{2}-2a(1-a)$ Substitute $b=1-a$.

$=2a^2-2a+\frac{1}{2}$

$=2\left(a-\frac{1}{2}\right)^2 \geqslant 0$ ☞ Changes to an equal sign when $a-\frac{1}{2}=0$.

Therefore, $\frac{1}{2} \geqslant 2ab$

Thus, LHS $=$ RHS when $a=b=\frac{1}{2}$ ☞ If $a-\frac{1}{2}=0$ and $a+b=1$, then $a=\frac{1}{2}$ and $b=\frac{1}{2}$.

(1) Given $a+b=1$, prove the inequality $a^2+b^2 \geqslant \frac{1}{2}$.

$a+b=1$, so $b=1-a$

$(\text{LHS})-(\text{RHS})=a^2+(1-a)^2-\frac{1}{2}$ Substitute $b=1-a$.

$=2a^2-2a+\frac{1}{2}$

$=2\left(a-\frac{1}{2}\right)^2 \geqslant 0$

Therefore, $a^2+b^2 \geqslant \frac{1}{2}$

Thus, LHS $=$ RHS when $a=b=\frac{1}{2}$ ☞ If $a-\frac{1}{2}=0$ and $a+b=1$, then $a=\frac{1}{2}$ and $b=\frac{1}{2}$.

J196b

2. Prove the following inequalities.

(1) Given $a+b=1$, prove $a^2+b^2>ab$.

$a+b=1$, so $b=1-a$

$$
\begin{aligned}
(\text{LHS})-(\text{RHS}) &= a^2+(1-a)^2-a(1-a) \\
&= 3a^2-3a+1 \\
&= 3\left(a-\frac{1}{2}\right)^2+\frac{1}{4} \geqq \frac{1}{4} > 0
\end{aligned}
$$

Substitute $b=1-a$.

Thus, $a^2+b^2>ab$

(2) Given $a+b=1$, prove $a+b>ab$.

$a+b=1$, so $b=1-a$

$$
\begin{aligned}
(\text{LHS})-(\text{RHS}) &= a+(1-a)-a(1-a) \\
&= a^2-a+1 \\
&= \left(a-\frac{1}{2}\right)^2+\frac{3}{4} \geqq \frac{3}{4} > 0
\end{aligned}
$$

Thus, $a+b>ab$

J 197a KUMON

Proof of Inequalities

1. Given that $a>0$ and $b>0$, prove the following inequalities.
   Then state the conditions for which the LHS equals the RHS.

Ex.

$$\frac{a+b}{2} \geqslant \sqrt{ab}$$

[Sol] $(\text{LHS})-(\text{RHS}) = \dfrac{a+b-2\sqrt{ab}}{2}$

$$= \frac{(\sqrt{a}\,)^2+(\sqrt{b}\,)^2-2\sqrt{ab}}{2}$$

$$= \frac{(\sqrt{a}-\sqrt{b}\,)^2}{2} \geqslant 0$$

☜ Changes to an equal sign when $\sqrt{a}-\sqrt{b}=0$.

Therefore, $\dfrac{a+b}{2} \geqslant \sqrt{ab}$

LHS $=$ RHS when $a=b$ ☜ If $\sqrt{a}-\sqrt{b}=0$, then $a=b$.

(1) $\sqrt{ab} \geqslant \dfrac{2ab}{a+b}$

$(\text{LHS})-(\text{RHS}) = \dfrac{\sqrt{ab}(a+b)-2ab}{a+b}$

$$= \frac{\sqrt{ab}(a+b-2\sqrt{ab}\,)}{a+b}$$

$$= \frac{\sqrt{ab}(\sqrt{a}-\sqrt{b}\,)^2}{a+b} \geqslant 0$$

Therefore, $\sqrt{ab} \geqslant \dfrac{2ab}{a+b}$

LHS $=$ RHS when $a=b$ ☜ If $\sqrt{a}-\sqrt{b}=0$, then $a=b$.

J197b

Given two positive numbers, a and b, $\frac{a+b}{2}$ is called the ***arithmetic mean***, and $\sqrt{ab}$ is called the ***geometric mean***.

Relationship between *arithmetic mean* and *geometric mean*

When $a>0$ and $b>0$, then $\frac{a+b}{2} \geqslant \sqrt{ab}$

LHS = RHS when $a=b$.

☞ $\begin{pmatrix} \textit{arithmetic} \\ \textit{mean} \end{pmatrix} \geqslant \begin{pmatrix} \textit{geometric} \\ \textit{mean} \end{pmatrix}$

You may use this relationship in the following proofs.

Ex.

Given that $x>0$, prove $x+\frac{1}{x} \geqslant 2$. Then state the conditions for which the LHS equals the RHS.

[Sol] $\frac{\text{LHS}}{2} = \frac{x+\frac{1}{x}}{2} \geqslant \sqrt{x \cdot \frac{1}{x}}$

☞ Use $\begin{pmatrix} \textit{arithmetic} \\ \textit{mean} \end{pmatrix} \geqslant \begin{pmatrix} \textit{geometric} \\ \textit{mean} \end{pmatrix}$ i.e. $\frac{a+b}{2} \geqslant \sqrt{ab}$, with x as a, and $\frac{1}{x}$ as b.

$\frac{x+\frac{1}{x}}{2} \geqslant 1$

Thus, $x+\frac{1}{x} \geqslant 2$

LHS = RHS when $x=1$

☞ From $x=\frac{1}{x}$.

2. Given that $x>0$ and $y>0$, prove $\frac{x}{y}+\frac{y}{x} \geqslant 2$. Then state the conditions for which the LHS equals the RHS.

[Sol] $\frac{\text{LHS}}{2} = \frac{\frac{x}{y}+\frac{y}{x}}{2} \geqslant \sqrt{\frac{x}{y} \cdot \frac{y}{x}}$

☞ Use $\frac{a+b}{2} \geqslant \sqrt{ab}$, with $\frac{x}{y}$ as a, and $\frac{y}{x}$ as b.

$\frac{\frac{x}{y}+\frac{y}{x}}{2} \geqslant 1$

Thus, $\frac{x}{y}+\frac{y}{x} \geqslant 2$

LHS = RHS when $x=y$

☞ From $\frac{x}{y}=\frac{y}{x}$.

J 198a KUMON

Proof of Inequalities

1. Prove the following inequalities when $a>0$. Then state the conditions for which the LHS equals the RHS.

(1) $a+\dfrac{9}{a}\geqslant 6$

$$\frac{\text{LHS}}{2}=\frac{a+\dfrac{9}{a}}{2}\geqslant\sqrt{a\cdot\frac{9}{a}}$$ ☜ Use $\begin{pmatrix}\textit{arithmetic}\\ \textit{mean}\end{pmatrix}\geqslant\begin{pmatrix}\textit{geometric}\\ \textit{mean}\end{pmatrix}$.

$$\frac{a+\dfrac{9}{a}}{2}\geqslant 3$$

Therefore, $a+\dfrac{9}{a}\geqslant 6$

LHS $=$ RHS when $a=3$ ☜ From $a=\dfrac{9}{a}$.

(2) $2a+\dfrac{1}{8a}\geqslant 1$

$$\frac{\text{LHS}}{2}=\frac{2a+\dfrac{1}{8a}}{2}\geqslant\sqrt{2a\cdot\frac{1}{8a}}$$ ☜ Use $\begin{pmatrix}\textit{arithmetic}\\ \textit{mean}\end{pmatrix}\geqslant\begin{pmatrix}\textit{geometric}\\ \textit{mean}\end{pmatrix}$.

$$\frac{2a+\dfrac{1}{8a}}{2}\geqslant\frac{1}{2}$$

Therefore, $2a+\dfrac{1}{8a}\geqslant 1$

LHS $=$ RHS when $a=\dfrac{1}{4}$ ☜ From $2a=\dfrac{1}{8a}$.

2. Prove the following inequalities when $a>0$ and $b>0$. Then state the conditions for which the LHS equals the RHS.

Ex.

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{a}\right)\geqslant 4$$

[Sol] LHS $=2+ab+\dfrac{1}{ab}$

$a>0$ and $b>0$, so $ab>0$ and $\dfrac{1}{ab}>0$

Therefore,

$$2+ab+\frac{1}{ab}\geqslant 2+2\sqrt{ab\cdot\frac{1}{ab}}$$
$$=2+2$$
$$=4$$

Use $\begin{pmatrix}\textit{arithmetic}\\ \textit{mean}\end{pmatrix}\geqslant\begin{pmatrix}\textit{geometric}\\ \textit{mean}\end{pmatrix}$ in the form $A+B\geqslant 2\sqrt{AB}$ with ab as A and $\dfrac{1}{ab}$ as B.

Thus, $\left(a+\dfrac{1}{b}\right)\left(b+\dfrac{1}{a}\right)\geqslant 4$

LHS $=$ RHS when $ab=1$

From $ab=\dfrac{1}{ab}$.

(1) $\left(a+\dfrac{2}{b}\right)\left(b+\dfrac{8}{a}\right)\geqslant 18$

LHS $=\left(a+\dfrac{2}{b}\right)\left(b+\dfrac{8}{a}\right)=10+ab+\dfrac{16}{ab}$

$a>0$ and $b>0$, so $ab>0$ and $\dfrac{16}{ab}>0$

Therefore,

$$10+ab+\frac{16}{ab}\geqslant 10+2\sqrt{ab\cdot\frac{16}{ab}}$$
$$=10+8$$
$$=18$$

Use $\begin{pmatrix}\textit{arithmetic}\\ \textit{mean}\end{pmatrix}\geqslant\begin{pmatrix}\textit{geometric}\\ \textit{mean}\end{pmatrix}$.

Thus, $\left(a+\dfrac{2}{b}\right)\left(b+\dfrac{8}{a}\right)\geqslant 18$

LHS $=$ RHS when $ab=4$

From $ab=\dfrac{16}{ab}$.

J 199a KUMON

Proof of Inequalities

1. Prove the following inequality when $a>0$ and $b>0$. Then state the conditions for which the LHS equals the RHS.

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geqslant 4$$

[Sol] LHS $= (a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = 2+\frac{a}{b}+\frac{b}{a}$

$a>0$ and $b>0$, so $\frac{a}{b}>0$ and $\frac{b}{a}>0$

Therefore,

$$2+\frac{a}{b}+\frac{b}{a} \geqslant 2+2\sqrt{\frac{a}{b}\cdot\frac{b}{a}}=4$$ ☜ Use $\begin{pmatrix}\textit{arithmetic}\\ \textit{mean}\end{pmatrix} \geqslant \begin{pmatrix}\textit{geometric}\\ \textit{mean}\end{pmatrix}$.

Thus, $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geqslant 4$

LHS $=$ RHS when $a=b$ ☜ From $\frac{a}{b}=\frac{b}{a}$.

2. Prove the following inequality when $a>0$, $b>0$, $c>0$ and $d>0$. Then state the conditions for which the LHS equals the RHS.

$$\left(\frac{b}{a}+\frac{d}{c}\right)\left(\frac{a}{b}+\frac{c}{d}\right) \geqslant 4$$

[Sol] LHS $= \left(\frac{b}{a}+\frac{d}{c}\right)\left(\frac{a}{b}+\frac{c}{d}\right) = 2+\frac{bc}{ad}+\frac{ad}{bc}$

$a>0$, $b>0$, $c>0$, $d>0$, so $\frac{bc}{ad}>0$ and $\frac{ad}{bc}>0$

Therefore,

$$2+\frac{bc}{ad}+\frac{ad}{bc} \geqslant 2+2\sqrt{\frac{bc}{ad}\cdot\frac{ad}{bc}}=4$$ ☜ Use $\begin{pmatrix}\textit{arithmetic}\\ \textit{mean}\end{pmatrix} \geqslant \begin{pmatrix}\textit{geometric}\\ \textit{mean}\end{pmatrix}$.

Thus, $\left(\frac{b}{a}+\frac{d}{c}\right)\left(\frac{a}{b}+\frac{c}{d}\right) \geqslant 4$

LHS $=$ RHS when $ad=bc$ ☜ From $\frac{bc}{ad}=\frac{ad}{bc}$.

J199b

Let's think about this!

We can prove that $\left(a+\dfrac{1}{b}\right)\left(b+\dfrac{1}{a}\right) \geqslant 4$ is the example on J198b, in another way, as follows.

From $a>0$ and $b>0$, $\dfrac{1}{a}>0$ and $\dfrac{1}{b}>0$

$a+\dfrac{1}{b} \geqslant 2\sqrt{a\cdot\dfrac{1}{b}} = 2\sqrt{\dfrac{a}{b}}$ and, from $a=\dfrac{1}{b}$, LHS = RHS when $ab = \boxed{1}$

$b+\dfrac{1}{a} \geqslant 2\sqrt{b\cdot\dfrac{1}{a}} = 2\sqrt{\dfrac{b}{a}}$ and, from $b=\dfrac{1}{a}$, LHS = RHS when $ab = \boxed{1}$

Multiplying the above inequalities gives,

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{a}\right) \geqslant 2\sqrt{\frac{a}{b}}\cdot 2\sqrt{\frac{b}{a}} = 4\sqrt{\frac{a}{b}\cdot\frac{b}{a}} = 4$$

Therefore, $\left(a+\dfrac{1}{b}\right)\left(b+\dfrac{1}{a}\right) \geqslant 4$ and LHS = RHS when $ab = \boxed{1}$

Let's consider if we can prove that $\left(a+\dfrac{1}{b}\right)\left(b+\dfrac{4}{a}\right) \geqslant 9$ using the method shown above.

$a+\dfrac{1}{b} \geqslant 2\sqrt{\dfrac{a}{b}}$ and, from $a=\dfrac{1}{b}$, LHS = RHS when $ab = \boxed{1}$

$b+\dfrac{4}{a} \geqslant 2\sqrt{b\cdot\dfrac{4}{a}} = 4\sqrt{\dfrac{b}{a}}$ and, from $b=\dfrac{4}{a}$, LHS = RHS when $ab = \boxed{4}$

If we multiply both sides of the above inequalities,

$$\left(a+\frac{1}{b}\right)\left(b+\frac{4}{a}\right) \geqslant 2\sqrt{\frac{a}{b}}\cdot 4\sqrt{\frac{b}{a}} = \boxed{8}$$

This is different from the original inequality. Therefore, we cannot prove the inequality using this method.

In this case, we should use the method shown on J198b, i.e. expand first, then use (***arithmetic mean***) $\geqslant$ (***geometric mean***).

Answers: 1, 1, 1, 1, 4, 8

J 200a KUMON

Proof of Inequalities

1. Given the proportion $\frac{x+y}{3}=\frac{y+z}{4}=\frac{z+x}{5}$, find the value of $\frac{xy+yz+zx}{x^2+y^2+z^2}$.

[Sol] Let $\frac{x+y}{3}=\frac{y+z}{4}=\frac{z+x}{5}=k$,

$$\begin{cases} x+y=3k & \cdots① \\ y+z=4k & \cdots② \\ z+x=5k & \cdots③ \end{cases}$$

From ①, ② and ③, $x=2k$, $y=k$, $z=3k$,

Substituting $x=2k$, $y=k$, $z=3k$,

$$\frac{xy+yz+zx}{x^2+y^2+z^2}=\frac{2k^2+3k^2+6k^2}{4k^2+k^2+9k^2}=\frac{11k^2}{14k^2}=\mathbf{\frac{11}{14}}$$

2. Given that $a>0$, $b>0$, prove that $(a+b)\left(\frac{1}{a}+\frac{4}{b}\right)\geqslant 9$. Then state the conditions for which the LHS equals the RHS.

[Sol] $(a+b)\left(\frac{1}{a}+\frac{4}{b}\right)=5+\frac{4a}{b}+\frac{b}{a}$

$a>0$, $b>0$, so $\frac{4a}{b}>0$ and $\frac{b}{a}>0$

$$5+\frac{4a}{b}+\frac{b}{a}\geqslant 5+2\sqrt{\frac{4a}{b}\cdot\frac{b}{a}}$$ ☜ Use $\left(\begin{matrix}\textit{arithmetic}\\ \textit{mean}\end{matrix}\right)\geqslant\left(\begin{matrix}\textit{geometric}\\ \textit{mean}\end{matrix}\right)$.

$$=5+4$$
$$=9$$

Therefore, $(a+b)\left(\frac{1}{a}+\frac{4}{b}\right)\geqslant 9$

LHS $=$ RHS when $2a=b$ ☜ From $\frac{4a}{b}=\frac{b}{a}$.

3. Prove that $a^2+b^2+c^2 \geqslant ab+bc+ca$. Then state the conditions for which the LHS equals the RHS.

[Sol] $$\begin{aligned}(\text{LHS})-(\text{RHS}) &= a^2+b^2+c^2-(ab+bc+ca) \\ &= a^2-a(b+c)+(b^2-bc+c^2) \\ &= \left(a-\frac{b+c}{2}\right)^2-\frac{(b+c)^2}{4}+b^2-bc+c^2 \\ &= \left(a-\frac{b+c}{2}\right)^2+\frac{3b^2-6bc+3c^2}{4} \\ &= \left(a-\frac{b+c}{2}\right)^2+\frac{3}{4}(b^2-2bc+c^2) \\ &= \left(a-\frac{b+c}{2}\right)^2+\frac{3}{4}(b-c)^2 \geqslant 0\end{aligned}$$

Thus, $a^2+b^2+c^2 \geqslant ab+bc+ca$

LHS $=$ RHS when $a-\dfrac{b+c}{2}=0$ and $b-c=0$, i.e. when $a=b=c$.

Let's think about this!

Alternative solution to prove the inequality $a^2+b^2+c^2 \geqslant ab+bc+ca$

[Sol]

$$\begin{aligned}&(\text{LHS})-(\text{RHS}) \\ &= a^2+b^2+c^2-(ab+bc+ca) \\ &= \frac{1}{2}[(a^2-2ab+b^2)+(\ \boldsymbol{b^2-2bc+c^2}\ )+(\ \boldsymbol{c^2-2ca+a^2}\ )] \\ &= \frac{1}{2}[(a-b)^2+(\ \boldsymbol{b-c}\ )^2+(\ \boldsymbol{c-a}\ )^2] \geqslant 0\end{aligned}$$

Thus, $a^2+b^2+c^2 \geqslant ab+bc+ca$

LHS $=$ RHS when $a-b=0$, $b-c=0$, $c-a=0$, which means when $a=b=c$.

Answers: $b^2-2bc+c^2$, $c^2-2ca+a^2$, $b-c$, $c-a$